ČVUT

ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ V PRAZE

Matematics I

Today lecture: Ing. Jan Valášek, Ph.

Winter semester 2022/2023

Matematics I – Lecture 1

Recommended literature:

1. J. Neustupa: Matematics I. CTU Publishing House, Prague, 1996.
2. Neustupa, J. and Kračmar, S.: Problems in Mathematics I, CTU Publishing House, Prague, 1999.
3. Keisler, H. J.: Elementary Calculus: An Infinitesimal Approach, 2nd edition, Prindle, Weber & Schmidt, 1986.
4. Calculus Volume I., Volume II., Volume III., provided by cnx.org/.
5. College algebra, provided by https://cnx.org/ .

These slides will be available at
*http : //marian.fsik.cvut.čz/~neustupa/M*1_Neu_lecture01*.pdf*

Syllabus

- Sets, statements and logic
- Linear algebra - operations with vectors, linear independency
- Matrices - Gaussian elimination, determinants, system of linear equations
- Sequences - basic properties, limits
- Functions - domain and range, basic properties, elementary functions
- Limits - finite and infinite limits of functions, properties, calculation
- Derivatives - definitions, properties, geometrical and physical meaning
- Application of derivatives, analysis of arbitrary function
- Integration - the Riemann and Newton integral and their connection
- Integration - integral of elementary functions, substitution and integration by parts, applications in probability

Important

- Follow information from www.muvs.cvut.cz/
- Read your official email !!!
- Check information in KOS system

I. Linear algebra

I.1. Vector space

Spaces R^n and E_n. Set of all ordered n-tuples of real numbers is denoted by R^n.

Elements of R^n *are called points in* R^n *or arithmetic vectors with* ***n*** *components:*

Notation of points in R^n :	$[1, 2]$, $[x_1, x_2]$, etc.	if $n = 2$,
	$[2, 0, 5]$, $[x_1, x_2, x_3]$, etc.	if $n = 3$,
	$[x_1, x_2, \ldots, x_n]$, etc.	for any n.

Let us define distance of two points X = $[x_1, x_2, \ldots, x_n]$, and Y = $[y_1, y_2, \ldots, y_n]$, as real number given by

$$d(X, Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2},$$

Then R^n becomes so called n-dimensional Euclidean space, denoted as E_n .

Examples: E_1 is line, E_2 is plane.

Arithmetic n-dimensional space. Let us define the sum of two arithmetic vectors $X = [x_1, x_2, \dots, x_n]$, and $Y = [y_1, y_2, \dots, y_n]$ by

$$[x_1, x_2, \dots, x_n] + [y_1, y_2, \dots, y_n] = [x_1 + y_1, x_2 + y_2, \dots, x_n + y_n]$$

And the product of arbitrary real number λ and arbitrary vector $X = [x_1, x_2, \dots, x_n]$,

$$\lambda \cdot [x_1, x_2, \dots, x_n] = [\lambda x_1, \lambda x_2, \dots, \lambda x_n].$$

R^n with these operations is called n-dimensional arithmetic space.

Vectors in E_2. In following we will deal with "free vectors". Explain differences.

Set of all vectors in E_2 will be denoted by $\mathbf{V}(E_2)$.

Vectors will be denoted by $\mathbf{u}$, $\mathbf{v}$, etc.

Every vector can be given by means of its coordinates. Coordinates are written in round brackets, e.g. $\mathbf{u} = (-2, 1)$, $\mathbf{x} = (x_1, x_2)$ etc.

For $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$ of $\mathbf{V}(\mathrm{E}_2)$ and $\lambda \in \mathrm{R}$ we define:

Sum of vectors $\mathbf{u}$, $\mathbf{v}$: $\quad \mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$,

Product of vector $\mathbf{u}$ *and number* λ: $\quad \lambda \cdot \mathbf{u} = (\lambda u_1, \lambda u_2)$.

It can be easily verified:

(a) For any vectors $\mathbf{u}, \mathbf{v} \in \mathbf{V}(\mathrm{E}_2)$ and any real number λ the result of $\mathbf{u} + \mathbf{v}$ as well as $\lambda \cdot \mathbf{u}$ belongs to $\mathbf{V}(\mathrm{E}_2)$. This fact is called that $\mathbf{V}(\mathrm{E}_2)$ is closed to operations of sum and product with a real number.

(b) For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}(\mathrm{E}_2)$ and any real numbers α, β it holds:

(b1) $\quad \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$,

(b2) $\quad (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$,

(b3) $\quad 1 \cdot \mathbf{u} = \mathbf{u}$,

(b4) $\quad \alpha \cdot (\beta \cdot \mathbf{u}) = (\alpha \cdot \beta) \cdot \mathbf{u}$,

(b5) $\quad \alpha \cdot (\mathbf{u} + \mathbf{v}) = \alpha \cdot \mathbf{u} + \alpha \cdot \mathbf{v}$,

(b6) $\quad (\alpha + \beta) \cdot \mathbf{u} = \alpha \cdot \mathbf{u} + \beta \cdot \mathbf{u}$.

(c) There exists zero vector $\mathbf{o} = (0, 0)$ with following property: For any $\mathbf{u} \in \mathbf{V}(E_2)$ it holds

$$\mathbf{u} + \mathbf{o} = \mathbf{u}.$$

(d) There exists for each $\mathbf{u} \in \mathbf{V}(E_2)$ such a vector $-\mathbf{u} \in \mathbf{V}(E_2)$ called opposite vector to vector $\mathbf{u}$) , that holds

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{o}.$$

We can define similarly $\mathbf{V}(E_3)$ and so on. Comment operations.

More examples of vector spaces.

Definition (vector space). Each non-empty set with defined operations of addition of vectors and multiplication of vectors by real number with properties a)-d) is called vector space.

Examples: $\mathbf{V}(E_2)$, $\mathbf{V}(E_3)$

Due to last definition we can deal further with general vector space $\mathbf{V}$.

Following theorems can be proved:

Theorem: (existence of zero element) *There is unique zero element in space* $\mathbf{V}$.

Theorem: (existence of opposite element) *Opposite vector* $(-\mathbf{u})$ *to vector* $\mathbf{u}$ *is in vector space* $\mathbf{V}$ *uniquely determined.*

Theorem: *For vector space* $\mathbf{V}$ *and vector* $\mathbf{u} \in \mathbf{V}$ *and real* α *following holds:*

1) $0 \cdot \mathbf{u} = \mathbf{o}$, 2) $(-1) \cdot \mathbf{u} = -\mathbf{u}$, 3) $\alpha \cdot \mathbf{o} = \mathbf{o}$.

Definition (linear combination): If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are group of n vectors from vect. space $\mathbf{V}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real numbers then result of

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \ldots + \alpha_n \mathbf{u}_n \qquad \text{(which belongs to vect. space } \mathbf{V}\text{)}$$

is called linear combination of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

Examples.

Definition (linear dependency of vectors).
Group of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is called linear dependent if there exist such n real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, where at least one of them is non-zero, and it holds

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \ldots + \alpha_n \mathbf{u}_n = \mathbf{o}.$$

Group of vectors, which are not linearly dependent, are called linearly independent. Let us denote it as LI.

Question : How to determine if given group of vectors is LD or LI?

Theorem. *If oné vector from group* $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ *from vector space* $\mathbf{V}$ *is equal to zero vector, then the group is LD.*

Proof.

In general:

Theorem: *If and only if it is possible to express one from the group of n vectors (n>1) as linear combination of the others, then group* $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ *is LD.*

Sketch of proof.

Remark: *If group of vectors* $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ *from vect. space* $\mathbf{V}$ *contains two identical vectors then it is LD.*