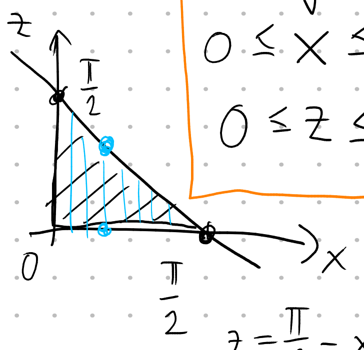
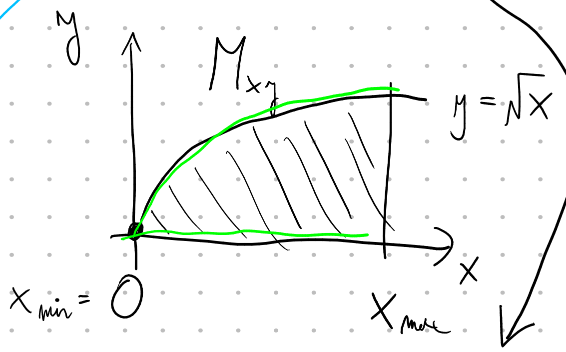
1.) $\underbrace{\iiint\limits_M y\cos(x+z)\,dx\,dy\,dz}_{=I}$ ∂M: $y=\sqrt{x}$ $y=0$ $\underbrace{x+z=\frac{\pi}{2} \quad z=0}$

sketch:

cut $y=0$



$0 \le y \le \sqrt{x}$
$0 \le x \le \frac{\pi}{2}$
$0 \le z \le \frac{\pi}{2} - x$



$z=0 \Rightarrow x_{max} = \frac{\pi}{2}$

$$I = \int_0^{\pi/2}\left(\int_0^{\sqrt{x}}\left(\int_0^{\pi/2-x} y\cos(x+z)\,dz\right)dy\right)dx =$$

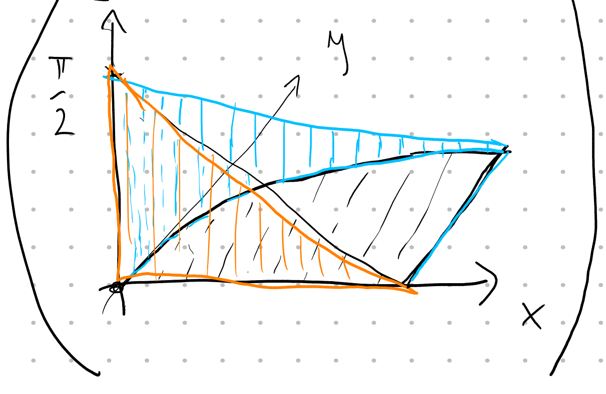
$$\left|\begin{array}{l|l} t = x+z & \frac{\pi}{2}-x \to \frac{\pi}{2} \\ dt = 0 + dz & 0 \to x \end{array}\right| = \int_0^{\pi/2}\left(\int_0^{\sqrt{x}}\left(\int_x^{\pi/2} y\cos t\,dt\right)dy\right)dx =$$

$$= \int_0^{\pi/2}\left(\int_0^{\sqrt{x}} \big[y\sin t\big]_x^{\pi/2}\,dy\right)dx = \int_0^{\pi/2}\int_0^{\sqrt{x}} y(1-\sin x)\,dy\,dx =$$

$$= \int_0^{\pi/2}\left[\frac{y^2}{2}(1-\sin x)\right]_0^{\sqrt{x}}dx = \frac{1}{2}\int_0^{\pi/2} x(1-\sin x)\,dx \underset{\text{by parts}}{=} \dots \ (M1)$$

$$= \frac{1}{2}\left(\frac{\pi^2}{8}-1\right) = \frac{\pi^2-8}{16}$$

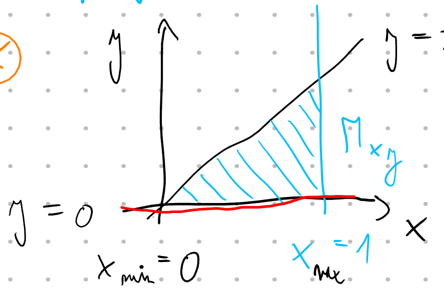
"3D sketch"



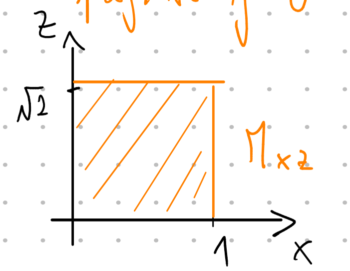
2) $\underbrace{\iiint\limits_M (x+y+z)\,dx\,dy\,dz}_{I}$

∂M: $x=1$, $y=x$, $y=0$, $z=0$, $z=\sqrt{2}$

proj. to $z=0$ (*)



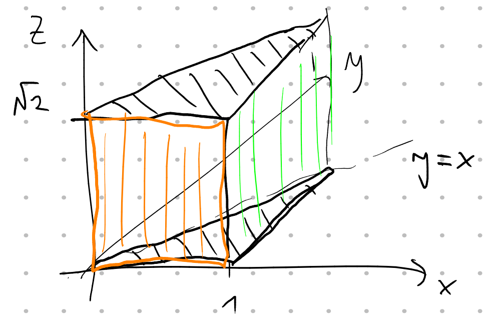
proj. to $y=0$ (*)



inequalities:

(*) $0 \le z \le \sqrt{2}$, $0 \le y \le x$, $0 \le x \le 1$ or (*) $0 \le y \le x$, $0 \le z \le \sqrt{2}$, $0 \le x \le 1$

1. way:



(*) $$I = \int_0^{\sqrt{2}}\int_0^1\int_0^x (x+y+z)\,dy\,dx\,dz =$$

$$= \int_0^{\sqrt{2}}\int_0^1 \left[xy + \frac{y^2}{2} + zy\right]_0^x dx\,dz =$$

$$= \underbrace{\int_0^{\sqrt{2}}\int_0^1 \left(\frac{3x^2}{2} + zx\right) dx\,dz}_{\text{note: } \iint\limits_{M_{xz}} dx\,dz} = \int_0^{\sqrt{2}} \left[\frac{x^3}{2} + z\frac{x^2}{2}\right]_0^1 dz =$$

$$= \int_0^{\sqrt{2}} \left(\frac{1}{2} + \frac{z}{2}\right) dz = \frac{1}{2}\left[z + \frac{z^2}{2}\right]_0^{\sqrt{2}} = \frac{1}{2}\left(\sqrt{2} + \frac{2}{2}\right) = \frac{1+\sqrt{2}}{2}$$

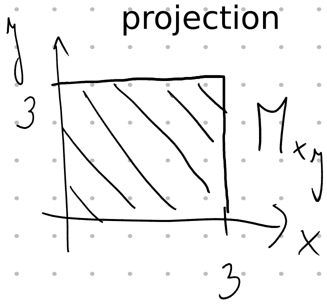
2. way:

(*) $$I = \int_0^1\int_0^x\int_0^{\sqrt{2}} (x+y+z)\,dz\,dy\,dx = \int_0^1\int_0^x \left[xz + yz + \frac{z^2}{2}\right]_0^{\sqrt{2}} dy\,dx =$$

$$= \underbrace{\int_0^1\int_0^x \left(x\sqrt{2} + y\sqrt{2} + 1\right) dy\,dx}_{\text{note: } \iint\limits_{M_{xy}} dx\,dy} = \int_0^1 \left[xy\sqrt{2} + \frac{y^2}{2}\sqrt{2} + y\right]_0^x dx =$$

$$= \int_0^1 \left(x^2\sqrt{2} + \frac{x^2}{2}\sqrt{2} + x\right) dx = \int_0^1 \left(\frac{3x^2\sqrt{2}}{2} + x\right) dx \overset{(M1)}{=} \dots = \frac{1+\sqrt{2}}{2}$$

projection



3.) $\rho(x,y,z) = x^2 + y^2$

$$m_{xy} = \iiint_M z\,\rho(x,y,z)\,dx\,dy\,dz =$$

$$= \int_0^3 \left(\int_0^3 \left(\int_0^{xy} z\,(x^2+y^2)\,dz \right) dy \right) dx =$$

$$= \iint_{M_{xy}} \left[\frac{z^2}{2}\right]_0^{xy} (x^2+y^2)\,dy\,dx = \iint_{M_{xy}} \frac{x^2y^2}{2}(x^2+y^2)\,dy\,dx =$$

$$= \frac{1}{2}\int_0^3 \left[x^4\frac{y^3}{3} + x^2\frac{y^5}{5}\right]_0^3 dx = \frac{1}{2}\int_0^3 \left(9x^4 + \frac{3^5}{5}x^2\right)dx =$$

$$= \frac{1}{2}\left[3^2\frac{x^5}{5} + \frac{3^5}{5}\frac{x^3}{3}\right]_0^3 = \frac{1}{2}\left(\frac{3^7}{5} + \frac{3^7}{5}\right) = \frac{3^7}{5}$$

$[kg \cdot m]$

4.) $\partial M: \quad z = \sqrt{3x^2+3y^2} = \sqrt{3}\sqrt{x^2+y^2} \quad /^2$

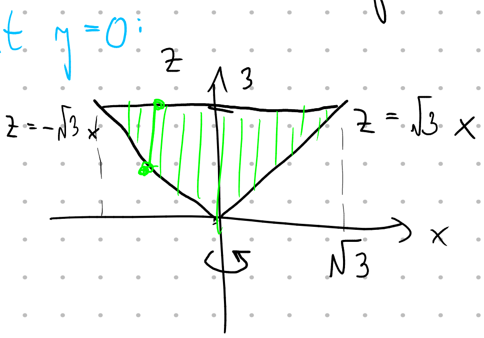
$$z^2 = 3x^2+3y^2$$

$$3x^2+3y^2-z^2=0 \quad \ldots \text{ conic surface}$$

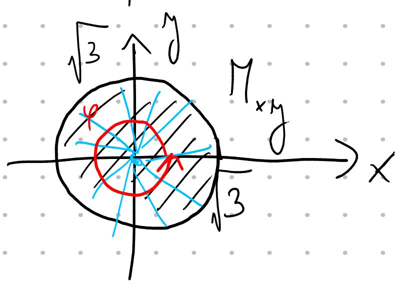
$$\boxed{x^2+y^2=r^2} \qquad 3r^2 - z^2 = 0$$

$$\sqrt{3}\, r = |z|$$

cut $y=0$:



$$\underbrace{\sqrt{3x^2+3y^2} \le z \le 3}$$



$$J_z = \iiint_M (x^2+y^2)\, \varrho(x,y,z)\, dx\,dy\,dz =$$

$$= \iint_{M_{xy}} \Big(\int_{\sqrt{3x^2+3y^2}}^{3} (x^2+y^2) \overset{\text{const.}}{\overset{\shortparallel}{\varrho}}\, dz \Big) dx\,dy = \iint_{M_{xy}} (x^2+y^2)\varrho\, [z]_{\sqrt{3x^2+3y^2}}^{3}\, dx\,dy =$$

subst. to polar coords.

$$= \varrho \iint_{M_{xy}} (x^2+y^2)\left(3 - \sqrt{3}\sqrt{x^2+y^2}\right) dx\,dy = \left| \begin{array}{l|l} x = r\cos\varphi & 0 \le r \le \sqrt{3} \\ y = r\sin\varphi & 0 \le \varphi \le 2\pi \\ J = r & \end{array} \right| =$$

$$= \varrho \int_0^{\sqrt{3}} \Big(\int_0^{2\pi} r^2 (3 - \sqrt{3}\, r)\, r\, d\varphi \Big) dr = \varrho \int_0^{\sqrt{3}} [\varphi]_0^{2\pi} (3r^3 - \sqrt{3}\, r^4)\, dr =$$

$$= 2\pi\varrho \left[\frac{3r^4}{4} - \frac{\sqrt{3}\, r^5}{5} \right]_0^{\sqrt{3}} = 2\pi\varrho \left(\frac{3^3}{4} - \frac{3^3}{5} \right) =$$

$$= 2\pi\varrho\, 3^3 \left(\frac{1}{4} - \frac{1}{5} \right) = \frac{27\pi\varrho}{10} \qquad [kg \cdot m^2]$$

5) $0 \leq z \leq 4 - \sqrt{x^2+y^2}$

$\partial M: \quad z = 4 - \sqrt{x^2+y^2}$

$(z-4)^2 = x^2+y^2$

$\underbrace{x^2+y^2}_{r^2} - (z-4)^2 = 0$

$r^{\cancel{2}} = (z-4)^{\cancel{2}}$

$f(x,y,z) = \sqrt{x^2+y^2}$

$\rightarrow$ cylindric coord.

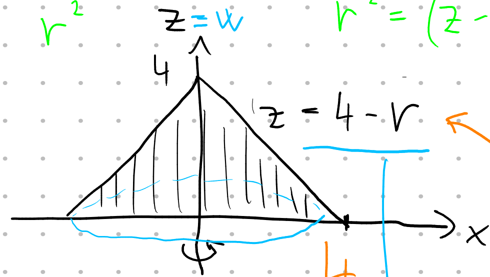
$$\boxed{\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ z &= w \\ \hline J &= r \end{aligned}}$$

$$\boxed{\begin{aligned} 0 \leq r \leq 4 \\ 0 \leq \varphi \leq 2\pi \end{aligned}}$$

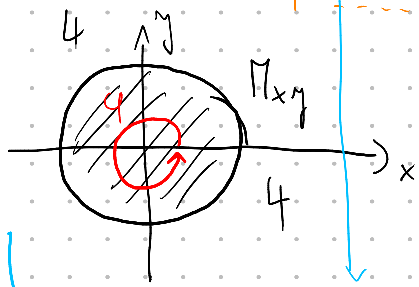
$0 \leq w \leq \cancel{4}$

not a cylinder!

$$\boxed{0 \leq w \leq 4 - r}$$



4 ..... $0 = 4 - r$



$w_{max} = 4 - r$

$$\iiint_M \overbrace{\sqrt{x^2+y^2}}^{r}\,dx\,dy\,dz = \int_0^4\left(\int_0^{2\pi}\left(\int_0^{4-r} r \cdot r \, dw\right)d\varphi\right)dr =$$

$$= \int_0^4\int_0^{2\pi} r^2 [w]_0^{4-r} \, d\varphi)\,dr = \int_0^4 [\varphi]_0^{2\pi} (4r^2 - r^3)\,dr =$$

$$= 2\pi \left[\frac{4r^3}{3} - \frac{r^4}{4}\right]_0^4 = 2\pi\left(\frac{2^8}{3} - \frac{2^8}{4}\right) = \frac{2^7\pi}{3}$$

c) $\text{mass} \stackrel{?}{=} \iiint_M \sqrt{x^2+y^2}\,dx\,dy\,dz$

$\text{mass} = \iiint_M \rho(x,y,z)\,dx\,dy\,dz$

✓

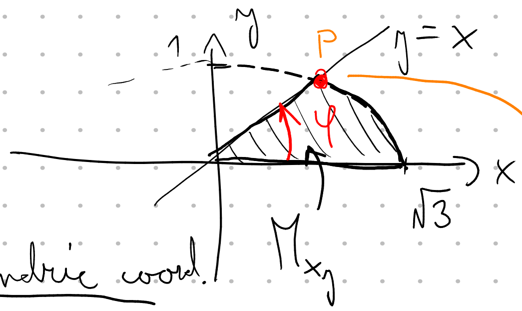
$\rho(x,y,z) = \sqrt{x^2+y^2} \geq 0$

6.) $0 \leq z \leq 1$

$0 \leq y \leq x$

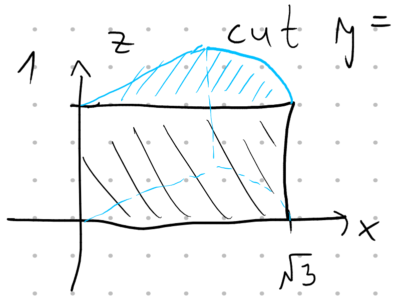
$\partial M \quad \frac{x^2}{3} + y^2 = 1 \quad \dashrightarrow$ elliptic cylinder

a)



"3D"

cut $y = 0$:



gen. cylindric coord.

$$\boxed{\begin{aligned} x &= r\sqrt{3}\cos\varphi \\ y &= r\sin\varphi \\ z &= w \\ J &= r\sqrt{3} \end{aligned}} \ast$$

note

$$\left(\begin{aligned} x &= r a \cos\varphi \qquad a = \sqrt{3} \\ y &= r b \sin\varphi \qquad b = 1 \end{aligned} \right)$$

boundaries:

$$0 \leq w \leq 1$$
$$0 \leq r \leq 1$$
$$0 \leq \varphi \leq \varphi_{max}$$

for point P:

$$x = y \wedge \frac{x^2}{3} + y^2 = 1$$

$$\frac{x^2}{3} + x^2 = 1$$

$$x^2 = \frac{3}{4} \Rightarrow x = \pm\frac{\sqrt{3}}{2}$$

$$x = R\sqrt{3}\cos\varphi \Rightarrow \overset{1}{R}\cancel{\sqrt{3}}\cos\varphi_{max} = \frac{\cancel{\sqrt{3}}}{2}$$

$$\cos\varphi_{max} = \frac{1}{2}$$

$$\varphi_{max} = \frac{\pi}{3}$$

!

(6a)

$$\iiint_M 1\,dx\,dy\,dz = \int_0^1\left(\int_0^{\pi/3}\left(\int_0^1 1\cdot r\sqrt{3}\,dr\right)d\varphi\right)dw =$$

b)

$$= \int_0^1\int_0^{\pi/3}\left[\frac{r^2}{2}\sqrt{3}\right]_0^1 d\varphi\,dw = \frac{\sqrt{3}}{2}\left[\varphi\right]_0^{\frac{\pi}{3}}\left[w\right]_0^1 =$$

$$= \frac{\pi\sqrt{3}}{6}$$

c) $V = \iiint_M 1\,dx\,dy\,dz$ (Volume)

$m = \iiint_M \rho(x,y,z)\,dx\,dy\,dz$ $\quad \rho(x,y,z) = 1$ (mass)

note:

$$* \quad J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \sqrt{3}\cos\varphi & -r\sqrt{3}\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} =$$

$$= r\sqrt{3}\cos^2\varphi + r\sqrt{3}\sin^2\varphi = r\sqrt{3}$$

7.) $C_z = \frac{m_{xy}}{m}$ $\qquad \rho(x,y,z) = \rho = \text{konst.}$

$$m_{xy} = \iiint_M z\rho \, dx\,dy\,dz = \int_0^h \int_0^{2\pi} \int_0^{h-r} w \rho r \, dw\, d\varphi\, dr$$

$$m = \iiint_M \rho \, dx\,dy\,dz = \int_0^h \int_0^{2\pi} \int_0^{h-r} \rho r \, dw\, d\varphi\, dr$$

$$\boxed{\begin{array}{lll} x = r\cos\varphi & & 0 \le r \le h \\ y = r\sin\varphi & & 0 \le \varphi \le 2\pi \\ z = w & , J = r & 0 \le w \le h-r \end{array}}$$

$$m_{xy} = \rho \int_0^h \int_0^{2\pi} \left[\frac{w^2}{2}\right]_0^{h-r} r\, d\varphi\, dr = \frac{\rho}{2} \int_0^h [\varphi]_0^{2\pi} (h^2 - 2hr + r^2) r\, dr =$$

$$= \pi\rho \int_0^h (h^2 r - 2hr^2 + r^3)\, dr = \pi\rho \left[h^2 \frac{r^2}{2} - 2h\frac{r^3}{3} + \frac{r^4}{4}\right]_0^h =$$

$$= \pi\rho \left(\frac{h^4}{2} - \frac{2h^4}{3} + \frac{h^4}{4}\right) = \pi\rho h^4 \frac{6-8+3}{12} = \frac{\pi\rho h^4}{12}$$

$$m = \rho \int_0^h \int_0^{2\pi} [w]_0^{h-r} r\, d\varphi\, dr = \rho \int_0^h [\varphi]_0^{2\pi} (h-r) r\, dr =$$

$$= 2\pi\rho \int_0^h (hr - r^2)\, dr = 2\pi\rho \left[h\frac{r^2}{2} - \frac{r^3}{3}\right]_0^h =$$

$$= 2\pi\rho \left(\frac{h^3}{2} - \frac{h^3}{3}\right) = 2\pi\rho h^3 \frac{3-2}{6} = \frac{\pi\rho h^3}{3}$$

$$C_z = \frac{m_{xy}}{m} = \frac{\frac{\cancel{\pi}\cancel{\rho} h^{\cancel{4}}}{\cancel{12}\ 4}}{\frac{\cancel{\pi}\cancel{\rho}\cancel{h^3}}{\cancel{3}}} = \frac{h}{4}$$