1.) $\underbrace{(A-\lambda I)}\vec{v}=\vec{0}$ $(\vec{v}\neq\vec{0})$

$$\begin{vmatrix} -2-\lambda & -2 & -9 \\ -1 & 1-\lambda & -3 \\ 1 & 1 & 4-\lambda \end{vmatrix}=0$$

$\vec{v}_{\lambda_2}=p\begin{pmatrix}3+2i\\1\\-1-i\end{pmatrix}$

$\vec{v}_{\lambda_3}=\overline{\vec{v}_{\lambda_2}}$ $p\in\mathbb{C}\setminus\{0\}$

$(-2-\lambda)(1-\lambda)(4-\lambda)+9+6+9(1-\lambda)+3(-2-\lambda)-2(4-\lambda)=0$

$(-2-\lambda)(1-\lambda)(4-\lambda)+9+\cancel{6}+9-9\lambda-\cancel{6}-3\lambda-8+2\lambda=0$

—||— $+10-10\lambda=0$

$(1-\lambda)((-2-\lambda)(4-\lambda)+10)=(1-\lambda)(-8-2\lambda+\lambda^2+10)=0$

$D=4-8=-4$

$\lambda_1=1$ ✓ $\lambda^2-2\lambda+2=0$

$\lambda_{2,3}=\frac{2\pm2i}{2}=1\pm i$

$\boxed{\lambda=1+i}$ $(A-(1+i)I)\vec{v}=\vec{0}$

$$\begin{pmatrix}-3-i & -2 & -9 & | & 0\\ -1 & -i & -3 & | & 0\\ 1 & 1 & 3-i & | & 0\end{pmatrix}\sim\begin{pmatrix}1 & 1 & 3-i & | & 0\\ -1 & -i & -3 & | & 0\\ 3+i & 2 & 9 & | & 0\end{pmatrix}\underset{\substack{I+II\\(3+i)I-III}}{\sim}\begin{pmatrix}1 & 1 & 3-i & | & 0\\ 0 & 1-i & -i & | & 0\\ 0 & 1+i & 1 & | & 0\end{pmatrix}\sim$$

$(3+i)(3-i)-9=9-i^2-9=1$

$i(1-i)=i-i^2=i+1$

$$\underset{i\,II-III}{\sim}\begin{array}{c}\begin{array}{ccc}x_1 & x_2 & x_3\end{array}\\ \begin{pmatrix}1 & 1 & 3-i & | & 0\\ 0 & 1-i & -i & | & 0\\ \cancel{0} & \cancel{0} & \cancel{0} & | & 0\end{pmatrix}\end{array}$$

$\vec{v}=(x_1,x_2,x_3)^T$

II: $(1-i)x_2-ix_3=0$

$x_2=p\in\mathbb{C}\setminus\{0\}$

$x_3=p\,\frac{1-i}{i}=p\left(\frac{i-i^2}{-1}\right)=-p(1+i)$

I: $x_1+x_2+(3-i)x_3=0$

$x_1=-x_2-(3-i)x_3=-p+(3-i)p(1+i)=p(\cancel{-1}+3+2i\cancel{+1})$

$=p(3+2i)$

2) a) $\underbrace{(A-\lambda I)}\vec{v}=\vec{0}$ $\lambda_1=-1$?

$$\begin{vmatrix} 2-\lambda & 1 & 2 \\ 5 & 2-\lambda & 6 \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2(-1-\lambda)+6-2(2-\lambda)+5(1+\lambda)=0$$

$$(2-(-1))^2\underbrace{\cancel{(-1+1)}}_{=0}+6-2(2-(-1))+0=6-2\cdot3=0 \checkmark$$

$\Longrightarrow \lambda_1$ is e.v.

other λ: $-(4-4\lambda+\lambda^2)(1+\lambda)+7+7\lambda=0$

$(1+\lambda)(-4+4\lambda-\lambda^2+7)=0$ $\longrightarrow$ $\lambda^2-4\lambda-3=0$

~~$(\lambda-3)(\lambda-1)=0$~~

$\lambda=\frac{4\pm\sqrt{28}}{2}$

$\lambda_1=-1$: $\underbrace{(A+I)}\vec{v}_{\lambda_1}=\vec{0}$ $\vec{v}=(x_1,x_2,x_3)^T$

$$\left(\begin{array}{ccc|c} 3 & 1 & 2 & 0 \\ 5 & 3 & 6 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 3 & 1 & 2 & 0 \\ 5 & 3 & 6 & 0 \end{array}\right) \begin{array}{l} \\ 3I-II \\ 5I-III \end{array} \sim \overbrace{\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 \\ \cancel{0} & \cancel{-3} & \cancel{-6} & \end{array}\right)}^{B}$$

$n=3$

$r(B)=2$

dim. sol. $=1$

$x_3=p\in\mathbb{R}-\{0\}$

II: $-x_2-2x_3=0$ $\rightarrow$ $x_2=-2p$

I $x_1=0$ $\vec{v}_{\lambda_1}=p\begin{pmatrix}0\\-2\\1\end{pmatrix}$

b) A^2 … $\lambda=?$ $\boxed{\lambda_{A^2}=\lambda_A^2}$ $\vec{v}_{\lambda A^2}=\vec{v}_{\lambda A}$

$\lambda_1=(-1)^2=1$ …… $\vec{v}_{\lambda_1,A^2}=\vec{v}_{\lambda_1,A}=p\begin{pmatrix}0\\-2\\1\end{pmatrix}$

$\lambda_{2,3A^2}=(1\pm\sqrt{7})^2=1\pm2\sqrt{7}+7=8\pm\sqrt{7}$

3.) a) $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

3 vec. in 2D

$$\underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \end{pmatrix}}_{A}$$

$r(A) < 3$:) $\Rightarrow$ LD (not LI)

b) $\alpha u + \beta v + \gamma w = 0$

$$\left(\vec{u} \;\; \vec{v} \;\; \vec{w} \;\middle|\; \vec{0}\right)$$ 1 sol. $\Rightarrow$ LI

$$\begin{vmatrix} a & 1 & 0 \\ 0 & 0 & a-1 \\ a+3 & 0 & a \end{vmatrix} \neq 0$$

$(a-1)(a+3) \neq 0$

$a \neq 1 \;\textcircled{v}\; a \neq -3$

$\boxed{\text{for } a \in \mathbb{R} - \{1, -3\}}$ Ⓛⓘ

$\Rightarrow$ u, v, w is base

c) $a = 1$ $b = \alpha_1 u + \alpha_2 v + \alpha_3 w$

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} \\ 4 & 0 & 1 \end{pmatrix}}_{B}\begin{pmatrix} 1 \\ \cancel{0} \\ -3 \end{pmatrix}^{(\vec{b})} \sim_{4\mathrm{I}-\mathrm{III}} \left(\begin{array}{ccc|c} \alpha_1 & \alpha_2 & \alpha_3 & \\ 1 & 1 & 0 & 1 \\ \cancel{0} & & & \\ 0 & 4 & -1 & -3 \end{array}\right)$$

3 unknowns

$r(B) = r(B|b) = 2$

$2 < 3 \Rightarrow \infty$

$3 - 2 = 1$ 1 param

$\alpha_2 = p \in \mathbb{R}$

III: $4\alpha_2 - \alpha_3 = -3 \Rightarrow \alpha_3 = 4p + 3$

I: $\alpha_1 + \alpha_2 = 1 \Rightarrow \alpha_1 = 1 - p$

$$\vec{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = p\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

E.g. $p = 0$

$\alpha_1 = 1$, $\alpha_3 = 3$

E.g. $p = 0$

$$\vec{b} = \vec{u} + 3\vec{w}$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & 1 \end{pmatrix}}_{C}\begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}^{(\vec{c})}$$

$r(C) \neq r(C|\vec{c}) \Rightarrow \nexists$ sol

$\vec{c} \neq \alpha_1 u + \alpha_2 v + \alpha_3 w$

4.) $$\begin{aligned} -x-2y+z&=3 \\ x+3y+az&=2a \\ 3x+5y&=a \end{aligned}$$

$A\vec{x}=\vec{b}$

$$\left(\begin{array}{ccc|c} \multicolumn{3}{c|}{\overbrace{}^{A}} & \overbrace{}^{b} \\ -1 & -2 & 1 & 3 \\ 1 & 3 & a & 2a \\ 3 & 5 & 0 & a \end{array}\right)$$

Gauss ?

$\det A \overset{?}{=} 0 : \begin{vmatrix} -1 & -2 & 1 \\ 1 & 3 & a \\ 3 & 5 & 0 \end{vmatrix} = 5-6a-9+5a=0$

$-a+4=0$

$\underline{\underline{a=-4}}$

for $\boxed{a\in\mathbb{R}-\{-4\}}$ $\det A\neq 0 \Rightarrow 1$ sol. (Cramer)

b) $\underline{a=-4}$

$$\left(\begin{array}{ccc|c} -1 & -2 & 1 & 3 \\ 1 & 3 & -4 & -8 \\ 3 & 5 & 0 & -4 \end{array}\right) \underset{\substack{I+II \\ 3I+III}}{\sim} \left(\begin{array}{ccc|c} -1 & -2 & 1 & 3 \\ 0 & 1 & -3 & -5 \\ 0 & -1 & 3 & 5 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} -1 & -2 & 1 & 3 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$r(A)=2=r(A|b)<\overset{n=3}{3}$

∞ sol.

$\rightarrow$ dim sol. $=3-2=1$

$\underline{\underline{z=p}}\in\mathbb{R}$

II $y-3z=-5 \Rightarrow y=\underline{\underline{-5+3p}}$

I: $-x-2y+z=3$

$x=-2y+z-3=+10-6p+p-3=\underline{7-5p}$

$$\underline{\underline{\vec{x}=p\begin{pmatrix}-5\\3\\1\end{pmatrix}+\begin{pmatrix}7\\-5\\0\end{pmatrix}}}$$

$$\det A^{-1} = \frac{1}{\det A}$$

5) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 4 & 5 \end{vmatrix} = 5 + 24 + 6 - 3 - 12 - 20 = 0$

$\nexists A^{-1}$ (A sing.)

6) $\begin{pmatrix} 2 & -1 & 3 \\ 0 & 5 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ 0 & 3 \\ 5 & 0 \end{pmatrix} =$

2×3 3×2 - 2×2

$$= \begin{pmatrix} 6+15 & 2-3 \\ 10 & 15 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 21 & -1 \\ 10 & 15 \end{pmatrix}}}$$

7.) $\begin{vmatrix} 1 & 0 & -1 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 \end{vmatrix} = 2 \cdot \overbrace{(-1)^5 \begin{vmatrix} 1 & -1 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{vmatrix}}^{A_{*3,2}} =$

$$= -2\,(1 + 1 - 1) = -2$$