Computer graphics
Lesson 3

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Coons cubic curve

- approximate curve, given points P_0, P_1, P_2 and P_3
- vector equation:

$$\mathbf{P}(t) = C_0(t)\mathbf{P}_0 + C_1(t)\mathbf{P}_1 + C_2(t)\mathbf{P}_2 + C_3(t)\mathbf{P}_3,\ t \in [0,1]$$

basis functions are Coons polynomials:

$$\begin{aligned}
C_0(t) &= \frac{1}{6}(1-t)^3, \\
C_1(t) &= \frac{1}{6}(3t^3 - 6t^2 + 4), \\
C_2(t) &= \frac{1}{6}(-3t^3 + 3t^2 + 3t + 1), \\
C_3(t) &= \frac{1}{6}t^3,
\end{aligned}$$

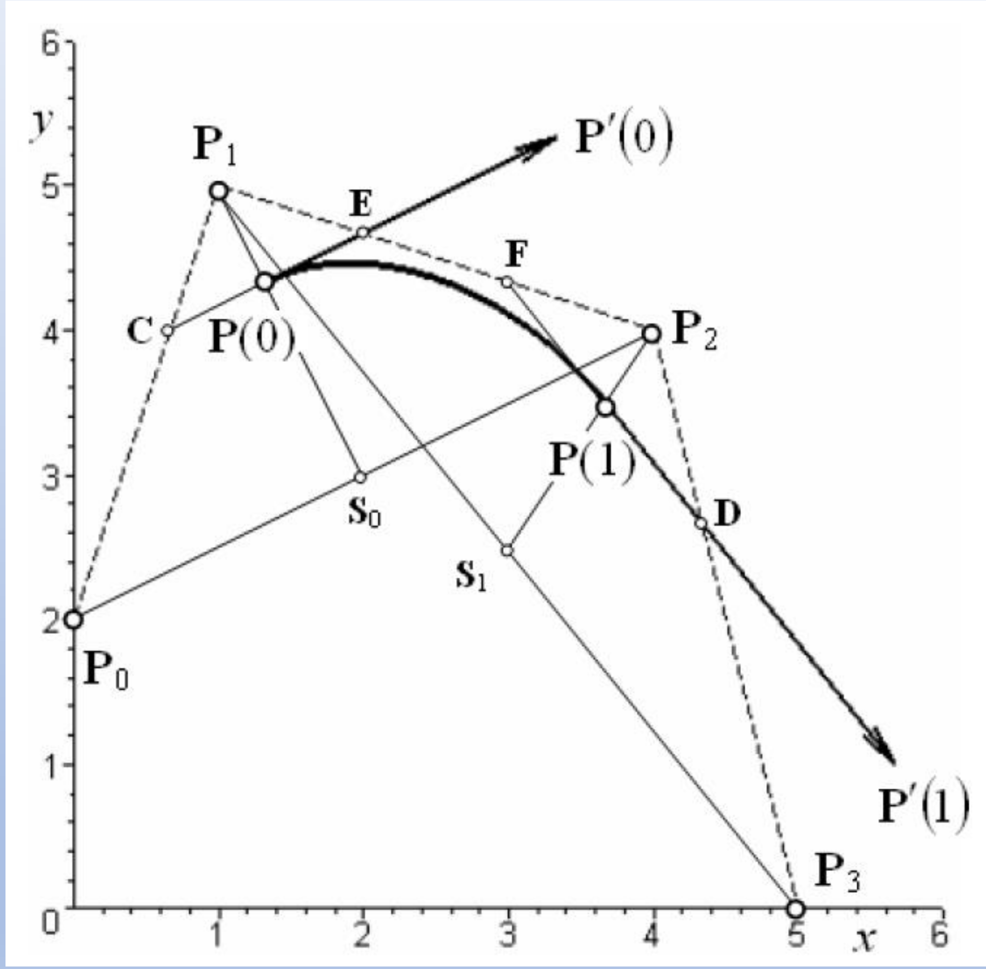
Coons cubic curve

- **properties:**
  1. values of Coons polynomials are <1 for any value of parameter *t* $\rightarrow$ curve does not pass through any given control point
  2. point $\mathrm{P}(0)$ lies at the „*anticentroid*“ of triangle $P_0P_1P_2$ constructed with respect to control point P_1
  3. point $\mathrm{P}(1)$ lies at the „*anticentroid*“ of triangle $P_1P_2P_3$ constructed with respect to control point P_2
  4. tangent vectors in $\mathrm{P}(0)$ and $\mathrm{P}(1)$ are given by equations:

$$\mathbf{P}'(0) = \frac{1}{2}\overrightarrow{\mathbf{P}_0\mathbf{P}_2} = \frac{1}{2}(\mathbf{P}_2 - \mathbf{P}_0)$$

$$\mathbf{P}'(1) = \frac{1}{2}\overrightarrow{\mathbf{P}_1\mathbf{P}_3} = \frac{1}{2}(\mathbf{P}_3 - \mathbf{P}_1)$$

Coons cubic curve



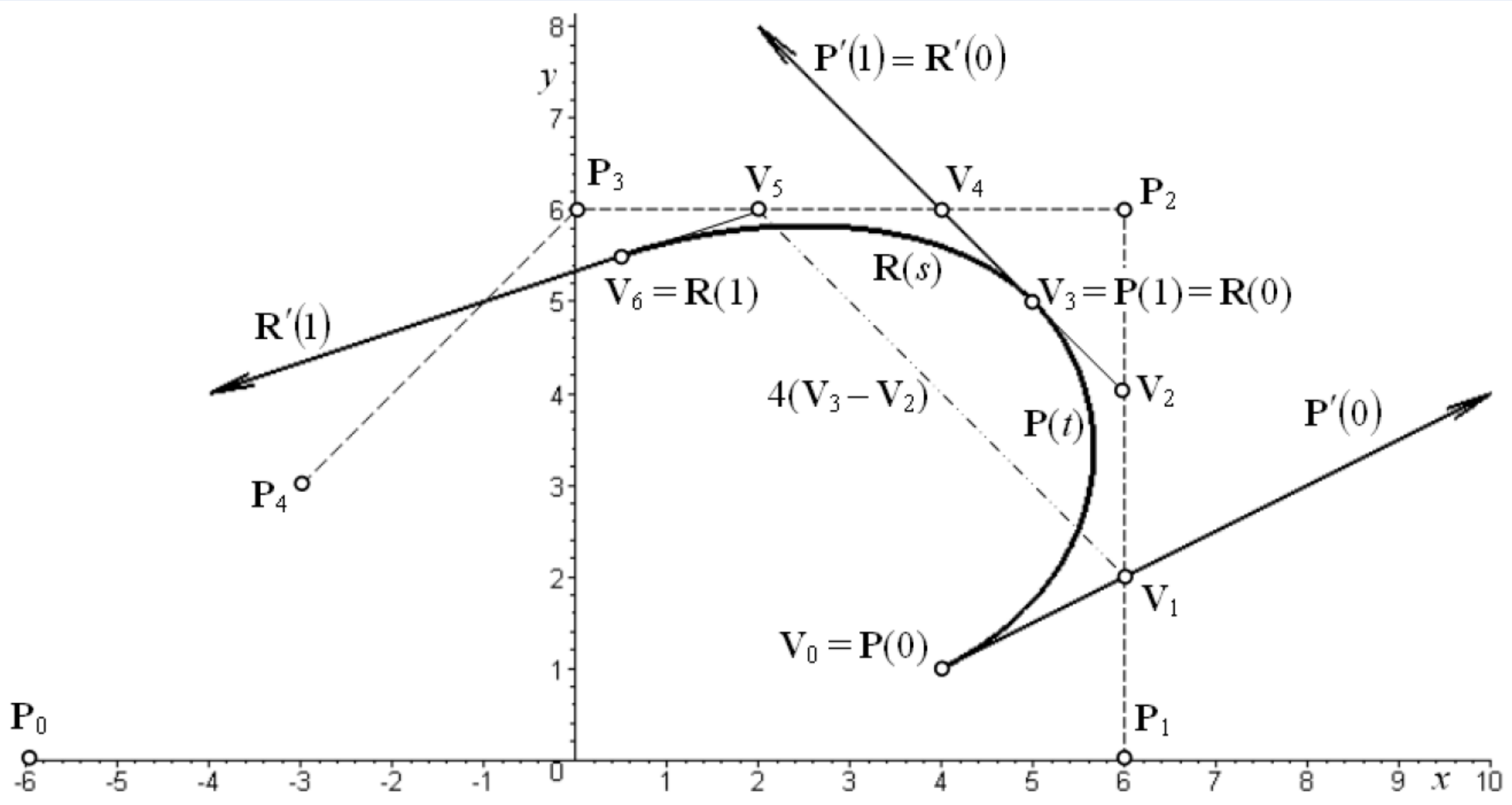
5. tangent line at the initial point intersects legs P_0P_1 and P_1P_2 at one third from control point P_1 (points *C*, *E*) and leg P_1P_2 intersects the tangent vector $P'(0)$ at one third from point $P(0)$ (point *E*)
6. tangent line at the terminal point intersects legs P_1P_2 and P_2P_3 at one third from control point P_2 (points F, D) and leg P_2P_3 intersects the tangent vector $P'(1)$ at one third from point $P(1)$ (point *D*)

Coons cubic curve

- **construction of endpoints:**
  1. divide all legs of control polygon in thirds to get points $0'$, 0, 0^*, 1, 1^* and $1'$
  2. construct straight line segments 00^* and 11^*
  3. initial point lies at the centre of straight line segment 00^*
  4. terminal point lies at the centre of straight line segment 11^*
  5. one half of vector 00^* is equal to one third of tangent vector at the initial point and one half of vector 11^* is equal to one third of tangent vector at the terminal point

Coons cubic B-spline

- piecewise C^2 continuous curve made of segments from Coons cubic curves with control points P_0, P_1, P_2, P_3 and P_1, P_2, P_3, P_4 etc.



Coons cubic B-spline

- given by a sequence of control points $P_0, P_1, \ldots, P_n, n \geq 4$ in space, a uniform B-spline curve of third degree R(t) compounded from *n-2* Coons cubic curves is called Coons cubic B-spline

$$\begin{aligned}
\mathbf{R}_0(t) &= C_0(t)\mathbf{P}_0 + C_1(t)\mathbf{P}_1 + C_2(t)\mathbf{P}_2 + C_3(t)\mathbf{P}_3, \ t \in [0,1], \\
\mathbf{R}_1(t) &= C_0(t)\mathbf{P}_1 + C_1(t)\mathbf{P}_2 + C_2(t)\mathbf{P}_3 + C_3(t)\mathbf{P}_4, \ t \in [0,1], \\
&\vdots \\
\mathbf{R}_{n-3}(t) &= C_0(t)\mathbf{P}_{n-3} + C_1(t)\mathbf{P}_{n-2} + C_2(t)\mathbf{P}_{n-1} + C_3(t)\mathbf{P}_n, \ t \in [0,1]
\end{aligned}$$

- closed or open

Coons cubic B-spline

- **properties:**
  - control polygon is created by at least five control points
  - If the last three control points are identical with the first three control points, i.e. $P_n = P_2$, $P_{n-1} = P_1$, $P_{n-2} = P_0$, Coons cubic B-spline is closed, otherwise it is open
  - does not pass through any control point of its control polygon
  - is created by *n-2* C^2 continuously joined Coons cubic curves, endpoints of these curves are called ***knots*** of Coons cubic B-spline
  - knots and tangent vectors at these knots can be constructed according to the properties of Coons cubic curve

Coons cubic B-spline

- **properties:**
  - is a piecewise defined curve by partially overlapping control polygons (a change of position of one control point does not cause the change of whole Coons cubic B-spline, it influences the shape of those individual Coons cubic curves, of which vector equation contains the changing control point)
  - the domain of each individual Coons cubic curve is $t \in [0; 1] \rightarrow$ the curve is called a ***uniform curve*** or ***curve with a uniform parametrization***

Coons cubic B-spline

Example 2.13 – Open and closed Coons cubic B-spline. The following control points of Coons cubic B-spline k are given:
$\mathbf{P}_0 = (0, 0)$, $\mathbf{P}_1 = (-6, 6)$, $\mathbf{P}_2 = (-6, 0)$,
$\mathbf{P}_3 = (0, -6)$, $\mathbf{P}_4 = (6, 0)$, $\mathbf{P}_5 = (6, 6)$, $\mathbf{P}_6 = \mathbf{P}_0 = (0, 0)$.

a) How many Coons cubic curves create this Coons cubic B-spline k? Write control polygons of individual Coons cubic curves. Draw a picture.

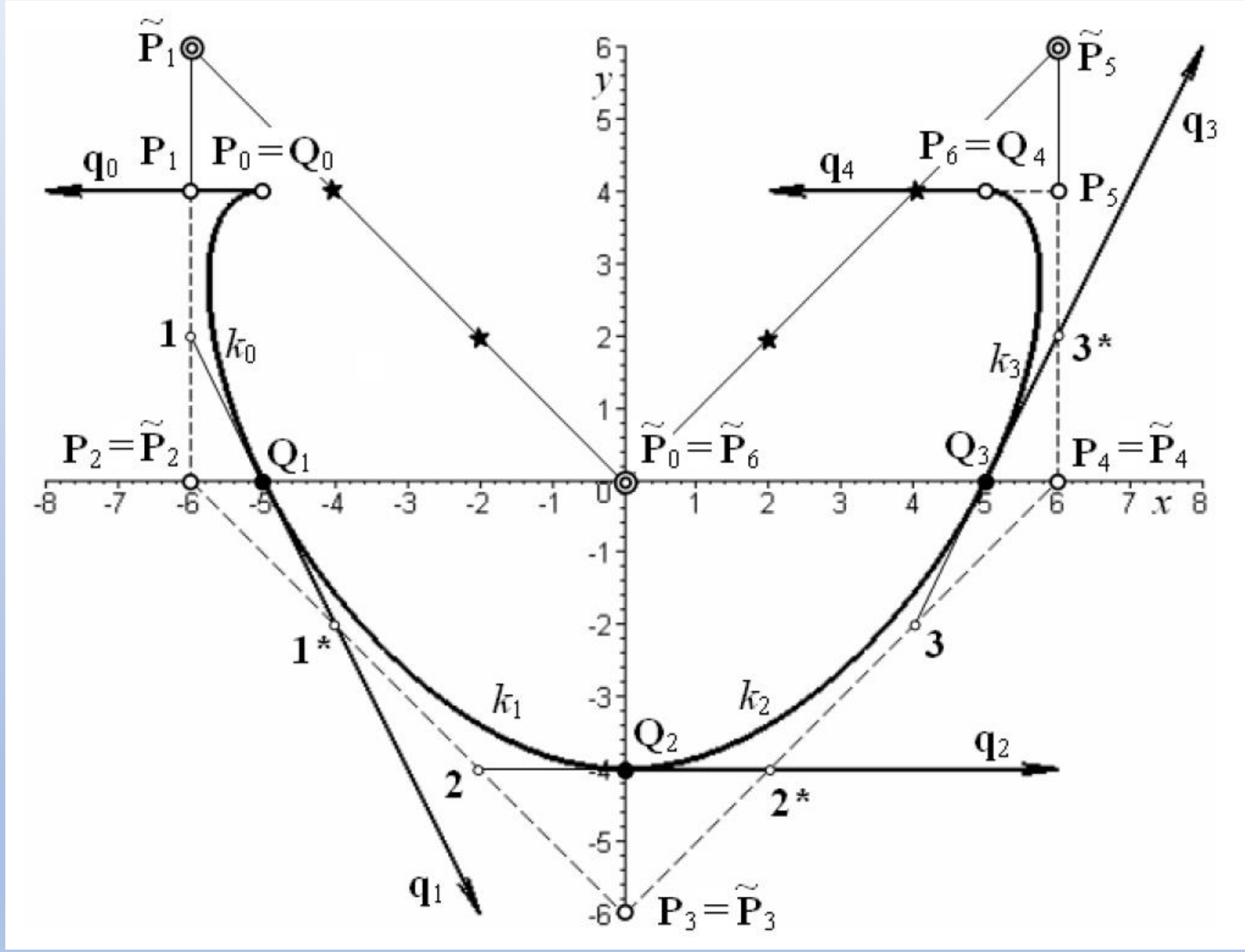
b) Modify the given control polygon of Coons cubic B-spline k to create a closed curve. Draw a picture.

Exercise 2.19 Coons cubic B-spline is given by control polygon
$\mathbf{P}_0 = (0, -6)$, $\mathbf{P}_1 = (-6, 0)$, $\mathbf{P}_2 = (-6, 6)$, $\mathbf{P}_3 = (6, 6)$, $\mathbf{P}_4 = (6, 0)$, $\mathbf{P}_5 = (12, 0)$.
Draw control points, construct control polygon, construct knots of Coons cubic B-spline and tangent vectors at these knots. Sketch Coons cubic B-spline. How many curve segments create Coons cubic B-spline? What is the continuity between individual curve segments of Coons cubic B-spline?

Clamped curve

= Uniform clamped B-spline curve of 3th degree

- segments are created by Bézier cubic curves/Coons cubic curves
- initial point is „anticentroid“ Q_0 of triangle $\tilde{P}_0\tilde{P}_1\tilde{P}_2$ constructed with respect to point $\tilde{P}_1$
- terminal point is „anticentroid“ Q_4 of triangle $\tilde{P}_4\tilde{P}_5\tilde{P}_6$ constructed with respect to point $\tilde{P}_5$
- P_1 is at the first third of tangent vector q_0, on the leg $\tilde{P}_1\tilde{P}_2$ at one third from point $\tilde{P}_1$
- P_1 is on the leg $\tilde{P}_4\tilde{P}_5$ at one third from point $\tilde{P}_4$



Clamped curve

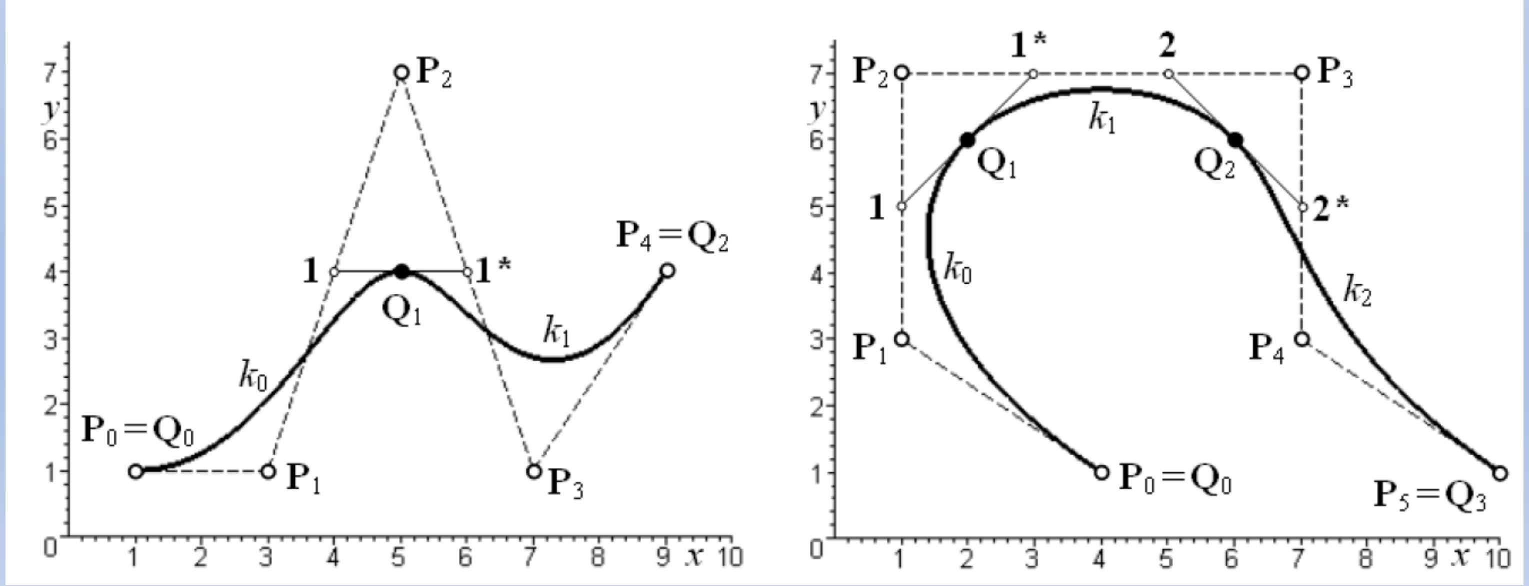
- transformation formulas between control points P_0 ,…, P_n of clamped curve and control points $\tilde{P}_0$,…, $\tilde{P}_n$ of open Coons cubic B-spline:

$$\begin{aligned}
\mathbf{P}_0 &= \mathbf{Q}_0 = \tfrac{1}{6}\widetilde{\mathbf{P}}_0 + \tfrac{2}{3}\widetilde{\mathbf{P}}_1 + \tfrac{1}{6}\widetilde{\mathbf{P}}_2, \\
\mathbf{P}_1 &= \tfrac{2}{3}\widetilde{\mathbf{P}}_1 + \tfrac{1}{3}\widetilde{\mathbf{P}}_2, \\
\mathbf{P}_i &= \widetilde{\mathbf{P}}_i, \ i = 2, \ldots, n-2, \\
\mathbf{P}_{n-1} &= \tfrac{1}{3}\widetilde{\mathbf{P}}_{n-2} + \tfrac{2}{3}\widetilde{\mathbf{P}}_{n-1}, \\
\mathbf{P}_n &= \mathbf{Q}_{n-2} = \tfrac{1}{6}\widetilde{\mathbf{P}}_{n-2} + \tfrac{2}{3}\widetilde{\mathbf{P}}_{n-1} + \tfrac{1}{6}\widetilde{\mathbf{P}}_n
\end{aligned}$$

- first 2 and the last 2 curve segments are created by Bézier curves and all inner curve segments are created by Coons cubic curves (Coons cubic B-spline)

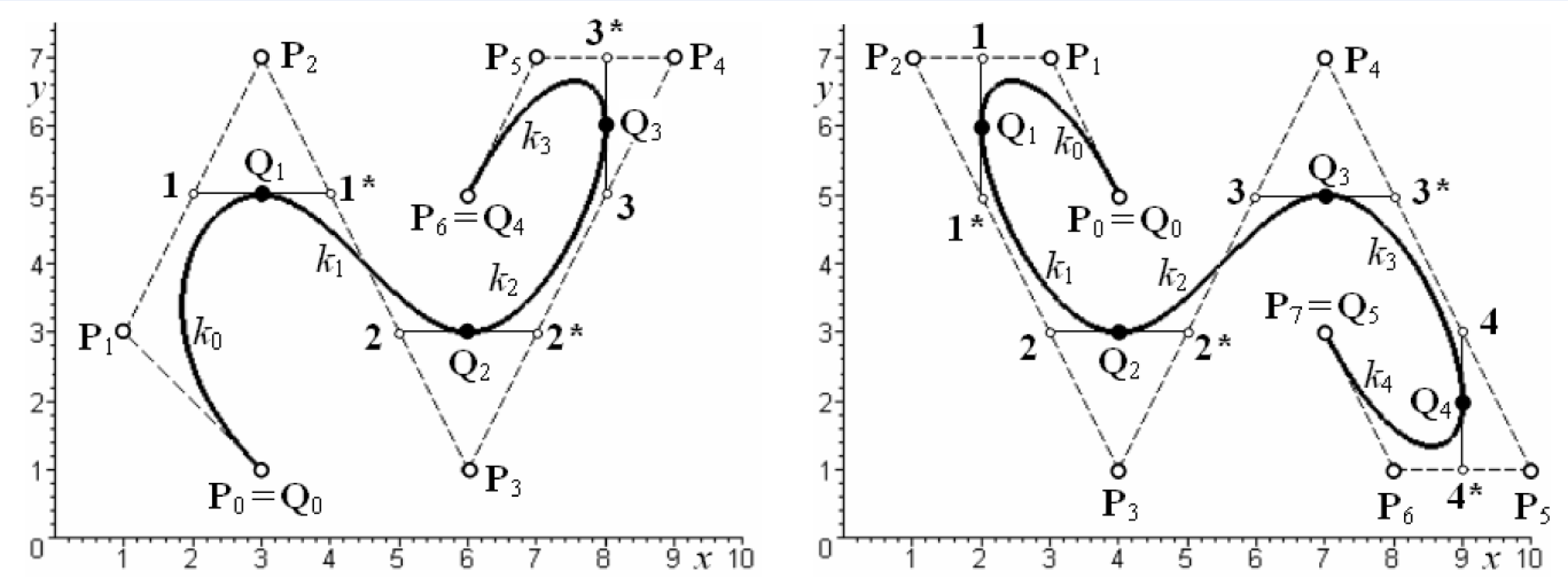
Clamped curve

- **properties:**
  - from properties of Bézier cubic curve, Coons cubic curve and Coons cubic B-spline
  - for n=3: only one curve segment → Bézier cubic curve
  - for n=4: two curve segments → Bézier cubic curves
  - for n=5: three curve segments → Bézier cubic curves



Clamped curve

- **properties:**
  - for n=6: four curve segments → Bézier cubic curves
  - for n=7: five curve segments → Bézier cubic curves, the middle curve segment is simultaneously Coons cubic curve



Clamped curve

- **properties:**
  - for n>7: n-2 curve segments → first two and the last two curve segments are Bézier cubic curves, remaining n-6 internal curve segments are C^2 continuously joined Coons cubic curves (open Coons cubic B-spline)
  - for n≥3 it is possible to create the clamped curve as a set of C^2 continuously joined Bézier cubic curves

Clamped curve

- **construction of knots of clamped curve for n>7:**
  1. initial point Q_0 is equal to the first control point P_0
  2. terminal point Q_{n-2} is equal to the last control point P_n
  3. do **not** divide the first and the last leg of control polygon
  4. divide the second and the next-to-last leg of the control polygon in halves → points 1 and $(n-3)^*$
  5. divide the remaining internal legs in thirds → points 1^*, 2, 2^*,…
  6. construct straight line segments 11^*, 22^*, …
  7. knots Q_1, Q_2, … lie at the centers of straight line segments 11^*, 22^*, …

Clamped curve

Exercise 2.26 Control polygons of clamped curves depicted in Fig. 2.26 are as follows:

a) $\mathbf{P}_0 = (1,1)$, $\mathbf{P}_1 = (3,1)$, $\mathbf{P}_2 = (5,7)$, $\mathbf{P}_3 = (7,1)$, $\mathbf{P}_4 = (9,4)$;
b) $\mathbf{P}_0 = (4,1)$, $\mathbf{P}_1 = (1,3)$, $\mathbf{P}_2 = (1,7)$, $\mathbf{P}_3 = (7,7)$, $\mathbf{P}_4 = (7,3)$, $\mathbf{P}_5 = (10,1)$;
c) $\mathbf{P}_0 = (3,1)$, $\mathbf{P}_1 = (1,3)$, $\mathbf{P}_2 = (3,7)$, $\mathbf{P}_3 = (6,1)$, $\mathbf{P}_4 = (9,7)$, $\mathbf{P}_5 = (7,7)$, $\mathbf{P}_6 = (6,5)$;
d) $\mathbf{P}_0 = (4,5)$, $\mathbf{P}_1 = (3,7)$, $\mathbf{P}_2 = (1,7)$, $\mathbf{P}_3 = (4,1)$, $\mathbf{P}_4 = (7,7)$, $\mathbf{P}_5 = (10,1)$, $\mathbf{P}_6 = (8,1)$, $\mathbf{P}_7 = (7,3)$.

Suppose that these clamped curves are created from Bézier cubic curves. Determine by construction and calculation the Cartesian coordinates of control points of all these Bézier cubic curves and find their vector equations. Using vector equations of Bézier cubic curves and verify C^2 continuity of clamped curves.