~~$\dot{x} = Ax$~~ nelinearni system!

~~Cvičení 4. 5.~~ 1)

$$\dot{x} = x^2 - y^2$$
$$\dot{y} = -2x(y+1)$$

$M=[1,0]$ poč. podm.

$\left(\partial_x = \frac{\partial}{\partial x}\right)$

a) $P(x,y) = x^2 - y^2$
$Q(x,y) = -2x(y+1)$

$\downarrow$: $\partial_x P = 2x$ $\quad \partial_y P = -2y$
$\partial_x Q = -2(y+1)$ $\quad \partial_y Q = -2x$

spoj. na $\mathbb{R}^2 \Rightarrow \exists!$ traj. pro $\forall$ b. $\in \mathbb{R}^2$

B.R. (singulárni body):

b) $P(x,y) = 0 \iff x^2 - y^2 = 0$
$Q(x,y) = 0 \iff -2x(y+1) = 0 \rightarrow x = 0 \vee y = -1$

z I rce: $x = 0 \Rightarrow y = 0$
$y = -1 \Rightarrow x = \pm 1$

$[-1,-1], [1,-1], [0,0]$

(nelin. system)

c) " $\frac{dy}{dx} = \frac{-2x(y+1)}{x^2 - y^2}$ " $\quad \boxed{\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}}$

$$dh = \underbrace{(x^2 - y^2)}\,dy + \underbrace{2x(y+1)}_{\frac{\partial h}{\partial x}}\,dx = 0$$

Exaktní rce?

$\frac{\partial^2 h}{\partial y \partial x} = 2x = \frac{\partial^2 h}{\partial x \partial y}$ ✓

(Cauchy–Riemannovy podm.)

↳ (jako hledání potenciálu)

$\frac{\partial h}{\partial y} = x^2 - y^2$

$\frac{\partial h}{\partial x} = 2x(y+1) \quad / \int dx$

$h = x^2(y+1) + K(y)$

$\frac{\partial h}{\partial y} = \cancel{x^2} + \boxed{K'(y)} = \cancel{x^2} - y^2$

$K(y) = -\frac{y^3}{3}$

$\left(K(y) = \int -y^2\,dy\right)$

traj.: $h = \boxed{x^2(y+1) - \frac{y^3}{3} = C}$

$M \in$ traj. $\quad 1 = C$

$$x^2(y+1) - \frac{y^3}{3} = 1$$

2.) $\dot{x} = (3x^2 - y^2)\,y^{-4}$

$\dot{y} = 2xy^{-3}$ $\quad x(0) = y(0) = 1$ ...M=[x(0); y(0)]=[1;1]

a) $\exists!$ traj. ~~-obl.~~ ?

b) B.R.

c) rce fáz. traj.

$\dot{x} = P(x,y) = (3x^2 - y^2)\,y^{-4}$ $\qquad \partial_x P = \frac{6x}{y^4}$ $\qquad \partial_y P = -\frac{12x^2}{y^5} + \frac{2}{y^3}$

$\dot{y} = Q(x,y) = 2xy^{-3}$ $\qquad \partial_x Q = \frac{2}{y^3}$ $\qquad \partial_y Q = -\frac{6x}{y^4}$

spoj. pro $y \neq 0$

$$\boxed{[x,y] \in \mathbb{R} \times (-\infty, 0) \vee \boxed{\mathbb{R} \times (0, \infty)}}$$

b) B.R.: $0 = 3x^2 - y^2$

$0 = 2x \Rightarrow x = 0$

~~$y = 0$~~ $\notin$ obl.

nemá B.R.

c) $\frac{dy}{dx} = \frac{Q}{P} = \frac{2xy^{-3}}{(3x^2 - y^2)\,y^{-4}}$

$$dh = \underbrace{\frac{2x}{y^3}}_{\partial_x h}\,dx - \underbrace{\frac{3x^2 - y^2}{y^4}}_{\partial_y h}\,dy = 0$$

exaktní rce?

$$\frac{\partial^2 h}{\partial x \partial y} = -\frac{6x}{y^4} \stackrel{?}{=} -\frac{6x}{y^4} = \frac{\partial^2 h}{\partial y \partial x} \checkmark$$

$\frac{\partial h}{\partial x} = \frac{2x}{y^3} \quad / \int dx \quad h = \frac{x^2}{y^3} + K(y) \quad / \frac{\partial}{\partial y}$

$\frac{\partial h}{\partial y} = -\frac{3x^2 - y^2}{y^4} = -3\frac{x^2}{y^4} + K'(y) \quad / \cdot y^4$

$\cancel{-3x^2} + y^2 + \cancel{3x^2} = K'(y)\,y^4$

$\frac{1}{y^2} = K'(y) \quad / \int dy$

$-\frac{1}{y} = K(y)$

$h = \frac{x^2}{y^3} - \frac{1}{y}$

1. integral pohybu

traj.:

$\frac{x^2}{y^3} - \frac{1}{y} = C$

"Cauchy" $1 - 1 = C \Rightarrow C = 0$

$\frac{x^2}{y^3} - \frac{1}{y} = 0$

3) poč. bod. [1;1]

$$\dot{x} = xe^y + \ln y = P(x,y)$$
$$\dot{y} = -2x - e^y = Q(x,y)$$

J:
$$\partial_x P = e^y \qquad \partial_y P = xe^y - \frac{1}{y}$$
$$\partial_x Q = -2 \qquad \partial_y Q = -e^y$$

spoj pro $y > 0$

$$\boxed{[x,y] \in \mathbb{R} \times (0,\infty)}$$

b) B.R.:

$$xe^y + \ln y = 0$$
$$+2x + e^y = 0 \Rightarrow x = -\frac{e^y}{2}$$

$$+\frac{1}{2}e^{2y} = +\ln y$$

rce pro B.R.

(nemá řešení v R)

c)
$$\frac{dx}{dy} = \frac{P}{Q} = \frac{xe^y + \ln y}{-2x - e^y}$$

$$dh = \underbrace{(xe^y + \ln y)}_{\partial_y h} dy + \underbrace{(2x + e^y)}_{\partial_x h} dx = 0$$

$$\frac{\partial^2 h}{\partial y \partial x} = e^y = \frac{\partial^2 h}{\partial x \partial y} \quad \checkmark \quad \text{exaktní rce}$$

$$\frac{\partial h}{\partial x} = 2x + e^y \quad \Big/ \int dx$$

$$h = x^2 + xe^y + K(y) \quad /\partial_y$$

$$\frac{\partial h}{\partial y} = \cancel{xe^y} + K'(y) = \cancel{xe^y} + \ln y \quad \Big/ \int dy$$

$$K(y) = \int \ln y \, dy \quad \left| \begin{matrix} \text{p.p.} & u' = 1 & u = y \\ & v = \ln y & v' = \frac{1}{y} \end{matrix} \right|$$

$$= y \ln y - \int 1 \, dy = y \ln y - y$$

$$h(x,y) = x^2 + xe^y + y(\ln y - 1) = C$$

trajektorie

"Cauchy"

$$1 + e - 1 = C$$
$$C = e$$

$x = x(t)$ $\quad J = \begin{pmatrix} 0 & 1 \\ -\cos x & 0 \end{pmatrix}$

4) $\ddot{x} + \sin x = 0$ $\qquad x(0) = \frac{\pi}{2}, \ \dot{x}(0) = 0$

a) $\dot{x} = y \qquad = P(x,y)$ $\qquad J: \ \partial_x P = 0 \quad \partial_y P = 1$

$\dot{y} = -\sin x \qquad = Q(x,y)$ $\qquad \partial_x Q = -\cos x \quad \partial_y Q = 0$

$(\wedge \ y(0) = \dot{x}(0) = 0)$ $\qquad$ spoj. pre $\mathbb{R}^2 \Rightarrow \exists!$ traj.

b) $\sin x = 0 \qquad$ B.R. $= [k\pi; 0] \ ; \ k \in \mathbb{Z}$

c) $\frac{dy}{dx} = \frac{-\sin x}{y}$ $\rightarrow$ $y\,dy + \sin x\,dx = 0 \quad / \int$

($y \neq 0 \Rightarrow$ $\Rightarrow \sin x = 0$ to nie je B.R.)

Traj.: $\underline{\underline{\frac{y^2}{2} - \cos x = C}}$ $\qquad$ Cauchy: $C = 0$

5.) $\ddot{x} - 9x + x^3 = 0$ $\qquad x(0) = 1, \ \dot{x}(0) = 0 = y(0)$

$\dot{x} = y \qquad = P(x,y)$ $\qquad J: \ \partial_x P = 0 \quad \partial_y P = 1$

$\dot{y} = 9x - x^3 \qquad = Q(x,y)$ $\qquad \partial_x Q = 9 - 3x^2 \quad \partial_y Q = 0$

spoj. v $\mathbb{R}^2$

singulárne B.B.

b) $0 = y$

$0 = 9x - x^3 \Rightarrow x(9 - x^2) = 0$ $\qquad [-3,0], [0,0], [3,0]$

c) $\frac{dy}{dx} = \frac{9x - x^3}{y} \Rightarrow y\,dy - (9x - x^3)\,dx = 0 \quad / \int$ $\qquad$ tak jednoduchú "trakktorú" rec

$\frac{y^2}{2} + \frac{x^4}{4} - \frac{9}{2}x^2 = C \quad / \cdot 4$

Traj.: $\underline{\underline{2y^2 + x^4 - 18x^2 = C}}$

$[1,0] \in$ traj.: $\quad 0 + 1 - 18 = C \qquad$ tj. $\underline{\underline{2y^2 + x^4 - 18x^2 = -17}}$