Numerical Simulation of Newtonian and Non-Newtonian Flows in Bypass

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Motivation

# Non-Newtonian Fluids

- Many common fluids are non-Newtonian:
  - paints
  - solutions of various polymers
  - food products

- Applications:
  - biomedicine
  - food industry
  - chemistry
  - glaciology
Non-Newtonian behavior

Main points of non-Newtonian behavior:

- the ability of the fluid to shear thin or shear thicken in shear flows
- the presence of non-zero normal stress differences in shear flows
- the ability of the fluid to yield stress
- the ability of the fluid to exhibit relaxation
- the ability of the fluid to creep

Figure: Non-Newtonian fluid behavior
Blood flow

- Blood is considered as a continuum
- Blood is non-Newtonian suspension of cells in plasma
- Experimental tests reveal that blood exhibits non-Newtonian phenomena such as shear thinning, creep and stress relaxation
- Only shear thinning is considered in this work
- It is reasonable to model it as a Newtonian fluid in greater vessels
- Can be described by conservation laws of mass and momentum
- The pulsatile character of blood flow is not considered as well as the elasticity of arterial walls
Arterial flow phenomena in atherosclerotic vessel

- separation
- recirculation
- secondary flow motion

Figure: Atherosclerotic vessel
Power-law fluids

\[ \tau(e) = 2 \nu_0 |e|^r e, \]

\( \tau \) is the stress tensor

\( e = (e_{ij}), \ i, j = 1, 2 \) is the strain tensor with components

\[ e_{11} = u_x, \ e_{12} = e_{21} = (v_x + u_y)/2, \ e_{22} = v_y \]

\( |e| \) denotes the Euclidean norm of a tensor

\( \nu_0 \) is a positive constant related to the limit of generalized viscosity \( \mu_g \)

\( r \) is a constant of the model
Navier-Stokes equations

- The generalized system of 2D Navier-Stokes equations and continuity equation for incompressible laminar flows in conservative dimensionless form

\[ \tilde{R} W_t + F^i_x + G^i_y = \frac{\tilde{R}}{\text{Re}} (F^v_x + G^v_y), \quad \tilde{R} = \text{diag}||0, 1, 1|| \]

- Inviscid fluxes

\[ F^i = \begin{pmatrix} u \\ u^2 + p \\ uv \end{pmatrix}, \quad G^i = \begin{pmatrix} v \\ uv \\ v^2 + p \end{pmatrix} \]

- Reynolds number in 2D defined as \( \text{Re} = dq_\infty / \nu \)

- Quantity \( q_\infty \) is a characteristic velocity (the speed of upstream flow)

- \( \nu = \eta / \rho \) is the kinematic viscosity

- \( d \) is a length scale (the width of the channel)

- \( W = (p, u, v)^T \) is the vector of solution
Navier-Stokes equations

The dimensionless quantities
- components of the velocity vector \( u = u^* / q_\infty, \) \( v = v^* / q_\infty \)
- pressure \( p = p^* / \rho q_\infty^2 \)

Newtonian viscous fluxes
\[
F^\nu = \begin{pmatrix} 0 \\ u_x \\ v_x \end{pmatrix}, \quad G^\nu = \begin{pmatrix} 0 \\ u_y \\ v_y \end{pmatrix}
\]

Non-Newtonian viscous fluxes
\[
F^\nu = \begin{pmatrix} 0 \\ 2|e^r| u_x \\ 2|e^r| (v_x + u_y) \end{pmatrix}, \quad G^\nu = \begin{pmatrix} 0 \\ 2|e^r| (u_y + v_x) \\ 2|e^r| v_y \end{pmatrix}
\]
The generalized system of 3D Navier-Stokes equations and continuity equation for incompressible laminar flows in conservative dimensionless form

\[ \tilde{R} W_t + F^i_x + G^i_y + H^i_z = \frac{\tilde{R}}{Re} (F^v_x + G^v_y + H^v_z), \quad \tilde{R} = \text{diag}||0, 1, 1|| \]

Inviscid and viscous fluxes

\[
F^i = \begin{pmatrix} u \\ u^2 + p \\ uv \\ uw \end{pmatrix}, \quad G^i = \begin{pmatrix} v \\ uv \\ v^2 + p \\ vw \end{pmatrix}, \quad H^i = \begin{pmatrix} w \\ uw \\ vw \\ w^2 + p \end{pmatrix}
\]

Reynolds number in 3D defined as \( Re = d_h q_\infty / \nu \)

Quantity \( q_\infty \) is a characteristic velocity (the speed of upstream flow)

\( \nu = \eta / \rho \) is the kinematic viscosity

\( d_h = 4S / O \) is the hydraulic diameter

\( S \) is the area section of the duct

\( O \) is the wetted perimeter
Navier-Stokes equations

The dimensionless quantities

- components of the velocity vector
  \[ u = u^*/q_\infty, \quad v = v^*/q_\infty, \quad w = w^*/q_\infty \]
- pressure \( p = p^*/\rho q_\infty^2 \)

Newtonian viscous fluxes

\[
F^v = \begin{pmatrix}
0 \\
u_x \\
v_x \\
w_x
\end{pmatrix}, \quad G^v = \begin{pmatrix}
0 \\
u_y \\
v_y \\
w_y
\end{pmatrix}, \quad H^v = \begin{pmatrix}
0 \\
u_z \\
v_z \\
w_z
\end{pmatrix}
\]
Boundary conditions

- At the inlet the Dirichlet boundary condition for the velocity components \((u, v) = (q_\infty, 0)\) is prescribed and the pressure \(p\) is computed by extrapolation from a domain.

- At the outlet the value of the pressure is prescribed by \(p = p_2\), where \(p_2\) is the dimensionless value of the pressure, that is lower then the initial value of the pressure at the inlet to ensure pressure gradient. The velocity components are extrapolated at the outlet.

- On the walls one considers the non-permeability and no-slip conditions for the velocity and the value of the pressure is taken from inside of the domain.
A steady state solution is considered

Use of the artificial compressibility method

The continuity equation is completed with the term $p_t/a^2$, where $a^2 > 0$. The pressure satisfies the artificial equation of state: $p = \rho/\delta$, in which $\rho$ is the artificial density, $\delta$ is the artificial compressibility, that is connected to the artificial speed of sound by relation $a = \delta^{-\frac{1}{2}}$

Governing equations has the form

$$W_t + F^i_x + G^i_y = \frac{\tilde{R}}{Re} \left( F^v_x + G^v_y \right), \text{ where } W = \left( \frac{p}{a^2}, u, v \right)^T$$

After rewriting the previous equation

$$W_t = -(\tilde{F}_x + \tilde{G}_y), \text{ where } \tilde{F} = F^i - \frac{1}{Re} F^v, \tilde{G} = G^i - \frac{1}{Re} G^v$$
Finite volume method 2D

- The system of equations is integrated over a finite volume $D_{ij}$

$$\int \int_{D_{ij}} W_t \, dx \, dy = - \int \int_{D_{ij}} \left( \tilde{F}_x + \tilde{G}_y \right) \, dx \, dy.$$

- Mean value and Green’s theorem are applied

$$W_t \bigg|_{ij} = - \frac{1}{\mu_{ij}} \int_{\partial D_{ij}} \tilde{F} \, dy - \tilde{G} \, dx, \quad \mu_{ij} = \int \int_{D_{ij}} \, dx \, dy.$$

- Velocity derivatives are computed using dual volume cells

- The integral on the right hand side is numerically approximated by

$$W_t \bigg|_{ij} = - \frac{1}{\mu_{ij}} \sum_{k=1}^{4} \tilde{F}_{ij,k} \Delta y_k - \tilde{G}_{ij,k} \Delta x_k.$$
Finite volume method 3D

- **Governing equations has the form**

\[ W_t + F^i_x + G^i_y + H^i_z = \frac{\tilde{R}}{Re} (F^v_x + G^v_y + H^v_z), \text{ where } W = (p/a^2, u, v, w)^T \]

- **After rewriting the previous equation**

\[ W_t = -(\tilde{F}_x + \tilde{G}_y + \tilde{H}_z), \text{ where } \tilde{F} = F^i - \frac{1}{Re} F^v, \quad \tilde{G} = G^i - \frac{1}{Re} G^v, \quad \tilde{H} = G^i - \frac{1}{Re} H^v \]

- **The system of equations is integrated over a finite volume** \( D_{ij} \)

\[ \int\int\int_{D_{ijk}} W_t \, dx \, dy = - \int\int\int_{D_{ij}} (\tilde{F}_x + \tilde{G}_y + \tilde{H}_z) \, dV. \]
Mean value and Green’s theorem are applied

\[ W_t |_{ijk} = -\frac{1}{\mu_{ijk}} \int_{\partial D_{ijk}} (\tilde{F}, \tilde{G}, \tilde{H})_l n_l^0 \Delta S_l = \frac{1}{\mu_{ijk}} \oint_{\partial D_{ijk}} (\tilde{F}, \tilde{G}, \tilde{H})_l n_l, \]

\[ \mu_{ijk} = \iiint_{D_{ijk}} dV. \]

\[ n_l = n_l^0 \Delta S_l, \] where \( n_l^0 \) is unit normal vector and \( \Delta S_l \) is a volume of a side of finite volume cell

Velocity derivatives are computed using dual volume cells

The integral on the right hand side is numerically approximated by

\[ W_t |_{ijk} = \frac{1}{\mu_{ijk}} \sum_{l=1}^{6} \left( \tilde{F}_{ijk,l} n_{1l} + \tilde{G}_{ijk,l} n_{2l} + \tilde{H}_{ijk,l} n_{3l} \right) \]
Finite volume method

- geometry of basic finite volume cell (2D)

- numerical approximation of spatial part

\[ W_t \big|_{ij} = -\frac{1}{\mu_{ij}} \sum_{k=1}^{4} \tilde{F}_k \Delta y_k - \tilde{G}_k \Delta x_k \]

- numerical approximation of derivatives in viscous fluxes

\[ u_x = \frac{1}{\mu_d} \frac{\partial}{\partial d} \int u dy \approx \sum_{m=1}^{4} u_m \Delta y_m \]

\[ u_y = -\frac{1}{\mu_d} \frac{\partial}{\partial d} \int u dx \approx -\sum_{m=1}^{4} u_m \Delta x_m \]

Figure: Basic finite volume cell
Finite volume method

- **Three-stage Runge-Kutta method**

\[
W_{i,j}^n = W_{i,j}^{(0)}
\]
\[
W_{i,j}^{(r)} = W_{i,j}^{(0)} - \alpha_r \Delta t \overline{RW}_{i,j}^{(r-1)}, \quad r = 1, \ldots, m
\]
\[
W_{i,j}^{n+1} = W_{i,j}^{(m)}, \quad m = 3,
\]
\[
\overline{RW}_{i,j}^{(r-1)} = \tilde{RW}_{i,j}^{(r-1)} - DW_{i,j}^n
\]
\[
\alpha_1 = 0.5, \quad \alpha_2 = 0.5, \quad \alpha_3 = 1.0
\]

\(DW_{i,j}^n\) is the artificial viscosity term of Jameson’s type

- **Form of residual (2D and 3D)**

\[
RW_{ij} = \frac{1}{\mu_{ij}} \sum_{k=1}^{4} \left( \tilde{F}_{ij,k}\Delta y_k - \tilde{G}_{ij,k}\Delta x_k \right),
\]
\[
RW_{ijk} = \frac{1}{\mu_{ijk}} \sum_{l=1}^{6} \left( \tilde{F}_{ijk,l}n_{1l} + \tilde{G}_{ijk,l}n_{2l} + \tilde{H}_{ijk,l}n_{3l} \right)
\]
Artificial viscosity

- artificial viscosity term (Jameson’s type, 2D)

\[
D^{(2)} W_{ij} = E \gamma_i (W_{i+1,j} - 2W_{i,j} + W_{i-1,j}) \\
+ E \gamma_j (W_{i,j+1} - 2W_{i,j} + W_{i,j-1})
\]

\[E = \text{diag} ||\epsilon_1, \epsilon_2, \epsilon_3||, \epsilon_1 \epsilon_2, \epsilon_3 \in \mathbb{R}, \gamma_i = \max(\gamma_{i1}, \gamma_{i2}), \gamma_j = \max(\gamma_{j1}, \gamma_{j2})\]

\[
\gamma_{i1} = \frac{|p_{i+1,j} - 2p_{i,j} + p_{i-1,j}|}{|p_{i+1,j} + 2p_{i,j} + p_{i-1,j}|}, \quad \gamma_{i2} = \frac{|p_{i,j} - 2p_{i-1,j} + p_{i-2,j}|}{|p_{i,j} + 2p_{i-1,j} + p_{i-2,j}|},
\]

\[
\gamma_{j1} = \frac{|p_{i,j+1} - 2p_{i,j} + p_{i,j-1}|}{|p_{i,j+1} + 2p_{i,j} + p_{i,j-1}|}, \quad \gamma_{j2} = \frac{|p_{i,j} - 2p_{i,j-1} + p_{i,j-2}|}{|p_{i,j} + 2p_{i,j-1} + p_{i,j-2}|}.
\]

- time step

\[
\Delta t = \min_{i,j,k} \frac{\text{CFL} \mu_{i,j}}{\rho_A \Delta y_k + \rho_B \Delta x_k + 2 \frac{\Delta x_k^2 + \Delta y_k^2}{\text{Re} \left( \frac{\Delta y_k^2 + \Delta x_k^2}{\mu_{i,j}} \right)}}
\]
Comparison of isolines of velocity, $Re = 500$

Re=500, Non-Newtonian

Re=500, Newtonian
Comparison of vector field of velocity, Re = 500

Re=500, Non-Newtonian

Re=500, Newtonian
Velocity magnitude profiles, Re=500

Velocity magnitude, x=5 bypass, channel
Re=500

- non-Newtonian
- Newtonian
Convergence, Re=500

rez u, channel, Re=500, non-Newtonian

rez u, bypass, Re-500, non-Newtonian

rez u, channel, Re=500, Newtonian

rez u, bypass, Re=500, Newtonian
Comparison of isolines of velocity, $Re = 200$

Re=200, Non-Newtonian

Re=200, Newtonian
Comparison of vector field of velocity, $Re = 200$

Re=200, Non-Newtonian

Re=200, Newtonian
Velocity magnitude profiles, Re=200

velocity magnitude, x=5
bypass, channel
Re=200

non_newtonian
Newtonian
3D angular bypass, Re=500

Re=500, main channel

Re=500
The outlet pressure is prescribed by sinus function in form:

\[ p_2 = p_{20}(1 + \alpha \sin 2\pi \omega t), \]

where \( \omega \) is a frequency and \( \alpha \) is an amplitude.
Unsteady flow in bypass, $Re=1000$

$Re=1000, t=103.2, p=4$

$Re=1000, t=104.8, p=4$

$Re=1000, t=106.4, p=4$
Unsteady flow in bypass, Re=500

Re=500, s=40, t=100.2
per=4

Re=500, s=40, t=100.8
per=4

Re=500, s=40, t=101.4
per=4
Unsteady flow in bypass, $Re=500$
Conclusion

- Numerical model of Newtonian and non-Newtonian flow was implemented
- The model was tested on different geometries and for different Reynolds numbers

Next steps

- Use of different non-Newtonian models
- Increasing mesh quality
- Extension of the model to 3D
- Unsteady computation