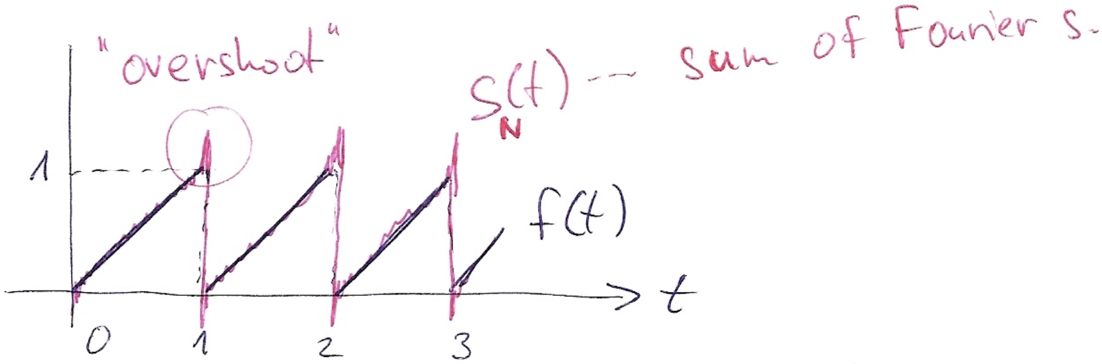
Remark: Gibbs Phenomenon

We have already seen, that the sum of Fourier series "oscillates" at "disconti-nuities", e.g.



It is possible to show, that
$\max(S_N(t)) = 1{,}0894\ldots$ for $N \to \infty$,
i.e. if we take the infinite number of terms in series, the overshoots does not disappear and they are "about 9%".
For more details see "Lecture notes for EE261, The Fourier Transform and its Applications, Brad Osgood, Stanford University".

Fourier integral

We will try to extend the concept of Fourier series to functions with period $\to \infty$, i.e. for non-periodic functions.

The Fourier series for function $f(t)$ with period L is

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{2\pi i k \frac{t}{L}}$$

with $$c_k = \frac{1}{L} \int_{-L/2}^{L/2} f(t) \, e^{-2\pi i k \frac{t}{L}} \, dt ,$$

let's denote $\dfrac{2\pi k}{L} = \omega_k$ ... the discrete angular velocity. The difference between the k-th and $(k+1)$-th angular velocity is

$$\omega_{k+1} - \omega_k = \frac{2\pi}{L}$$

$\rightarrow$ we see that $\lim\limits_{L \to \infty} (\omega_{k+1} - \omega_k) = 0$

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we know that

$$f(t) = \frac{\cancel{L}}{2\pi} \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{\cancel{L}} \int_{-L/2}^{L/2} f(t) e^{-i\omega_k t} dt}_{= g(\omega_k)} \, e^{i\omega_k t} \cdot \frac{2\pi}{L} =$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \underbrace{\underbrace{g(\omega_k) e^{i\omega_k t}}_{\text{some function at } \omega=\omega_k} \cdot \underbrace{(\omega_{k+1} - \omega_k)}_{\text{width of interval } \langle \omega_k, \omega_{k+1} \rangle}}_{\text{this is for } L \to \infty \text{ the definition of integral}}$$

this is for $L \to \infty$ the definition of integral

$$\int_{\omega_{-\infty}}^{\omega_{\infty}} g(\omega) e^{i\omega t} d\omega \, ,$$

(since $\omega_{k+1} - \omega_k \to 0$ for $L \to \infty$)

and $\omega_{-\infty} = -\infty$, $\omega_{\infty} = \infty$

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$$\Rightarrow f(t) = \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \right) e^{i\omega t} d\omega}_{\text{Fourier integral}}$$

Existence of Fourier integral:

The Fourier integral exists for function $f(t)$ which satisfies :

1) $f(t)$ is piecewise continuous for $t \in (-\infty, \infty)$ and $f(t)$ have bounded left and right limits in discontinuities

2) $\int_{-\infty}^{\infty} |f(t)| \, dt$ exists. $\left(\Rightarrow \underline{\underline{\lim_{t \to \pm\infty} f(t) = 0}} \right)$

The Fourier integral defines the Fourier transform :

$$\boxed{\begin{aligned} \mathcal{F}\{f\} &= F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \\ \mathcal{F}^{-1}\{F\} &= f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega \end{aligned}} \qquad \text{(FT)}$$

Fourier and inverse Fourier transform

<u>Remarks</u>: You can find different definitions of Laplace transform "due to the term $\frac{1}{2\pi}$", which can be placed in "different possitions". The above definition corresponds to the definition of Fourier transform in Maple.

<u>Remark</u>: Alternative form of Fourier integral.

If we follow the procedure starting at page 146, but with discrete frequency $\xi_k = \frac{k}{L}$ and $\xi_{k+1} - \xi_k = \frac{1}{L}$

then we can write

$$f(t) = \sum_{k=-\infty}^{\infty} \frac{1}{L} \underbrace{\int_{-L/2}^{L/2} f(t)\, e^{-2\pi \xi_k i t}\, dt}_{= g(\xi_k)} \; e^{2\pi \xi_k i t} =$$

$$= \sum_{k=\infty}^{\infty} g(\xi_k)\, e^{2\pi \xi_k i t} \cdot (\xi_{k+1} - \xi_k)$$

and in limit case $L \to \infty$ we get

$$\boxed{\begin{aligned} \hat{f}(\xi) &= \int_{-\infty}^{\infty} f(t)\, e^{-i 2\pi \xi t}\, dt \\ f(t) &= \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{i 2\pi \xi t}\, dt \end{aligned}}$$

which is also the definition of Fourier transform, where t can be "time" and then ξ is "frequency". $\hat{f}(\xi)$ corresponds to c_k at page 107 and according to page 123 the amplitude is $A_k = \sqrt{a_k^2 + b_k^2}$ and

$$a_k = c_k + c_{-k} \quad \text{and} \quad b_k = i(c_k - c_{-k})$$

$$\Rightarrow \underline{A_k = 2|c_k|} = 2 c_k \overline{c_k}$$

Therefore $2|\hat{f}(\xi)|$ should correspond to the amplitude spectrum of signal $f(t)$.

We will further use the definition of Fourier transform (FT) from page 149. It is possible to show that $\hat{f}(\xi) = F(\underbrace{2\pi\xi}_{=\omega})$.

<u>Example:</u> Let's try to compute Fourier transforms (FT) and amplitude for given functions

a) $f(t) = \sin t$

b) $g(t) = \sin t + \cos(10 \cdot t)$

c) $h_1(t) = e^{-t^2} \cos t$

d) $h_2(t) = e^{-t^2} \sin t$

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```
> restart:
> with(inttrans):
> f(t):=sin(t);
```

$$f(t) := \sin(t)$$

```
> F(w):=fourier(f(t),t,w);
```

$$F(w) := I\,\pi\,(\mathrm{Dirac}(w+1) - \mathrm{Dirac}(w-1))$$

```
> 2*abs(F(w))/2/Pi;
```

$$|-\mathrm{Dirac}(w+1) + \mathrm{Dirac}(w-1)|$$

here we see, that amplitude = $= \frac{2|F(\omega)|}{2\pi}$

```
> g(t):=sin(t)+cos(10*t);
```

$$g(t) := \sin(t) + \cos(10\,t)$$

```
> G(w):=fourier(g(t),t,w);
```

$$G(w) := \pi\,(I\,\mathrm{Dirac}(w+1) + \mathrm{Dirac}(w+10) - I\,\mathrm{Dirac}(w-1) + \mathrm{Dirac}(w-10))$$

```
> 2*abs(G(w))/2/Pi;
```

$$|-I\,\mathrm{Dirac}(w+1) - \mathrm{Dirac}(w+10) + I\,\mathrm{Dirac}(w-1) - \mathrm{Dirac}(w-10)|$$

amplitude is again $\frac{2|G(\omega)|}{2\pi}$

```
> h1(t):=exp(-t*t)*cos(t);   h2(t):=exp(-t*t)*sin(t);
```

$$h1(t) := \mathrm{e}^{(-t^2)}\cos(t)$$

$$h2(t) := \mathrm{e}^{(-t^2)}\sin(t)$$

```
> H1(w):=fourier(h1(t),t,w);   H2(w):=fourier(h2(t),t,w);
```

$$H1(w) := \sqrt{\pi}\,\cosh\left(\frac{1}{2}\,w\right)\,\mathrm{e}^{\left(-\frac{1}{4}\,w^2 - \frac{1}{4}\right)}$$

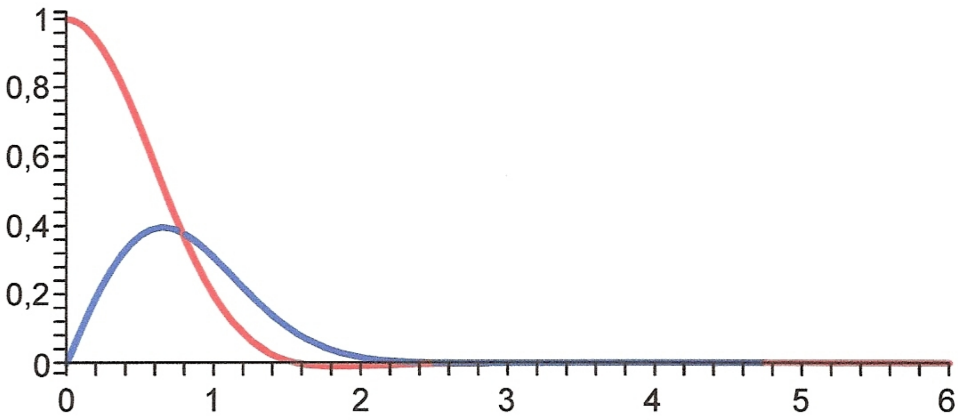
$$H2(w) := -I\,\sqrt{\pi}\,\sinh\left(\frac{1}{2}\,w\right)\,\mathrm{e}^{\left(-\frac{1}{4}\,w^2 - \frac{1}{4}\right)}$$

```
> A1(w):=2*abs(H1(w))/2/Pi;   A2(w):=2*abs(H2(w))/2/Pi;
```

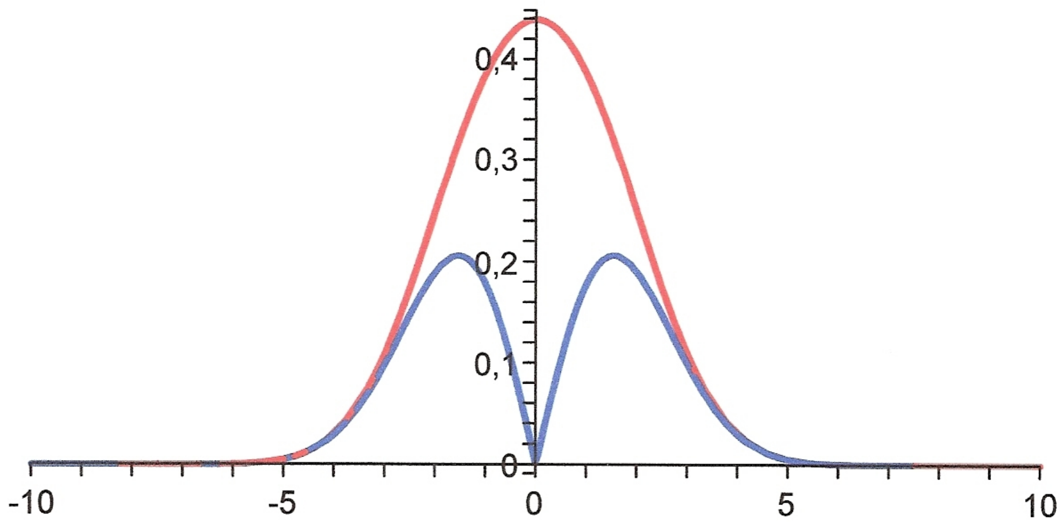
$$A1(w) := \frac{\mathrm{e}^{\left(-\frac{1}{4} - \frac{1}{4}\,\Re(w^2)\right)}\left|\cosh\left(\frac{1}{2}\,w\right)\right|}{\sqrt{\pi}}$$

$$A2(w) := \frac{\mathrm{e}^{\left(-\frac{1}{4} - \frac{1}{4}\,\Re(w^2)\right)}\left|\sinh\left(\frac{1}{2}\,w\right)\right|}{\sqrt{\pi}}$$

```
> with(plots):
Warning, the name changecoords has been redefined
> p1:=plot(h1(t),t=0..6,color=red):   p2:=plot(h2(t),t=0..6,color=blue):
```



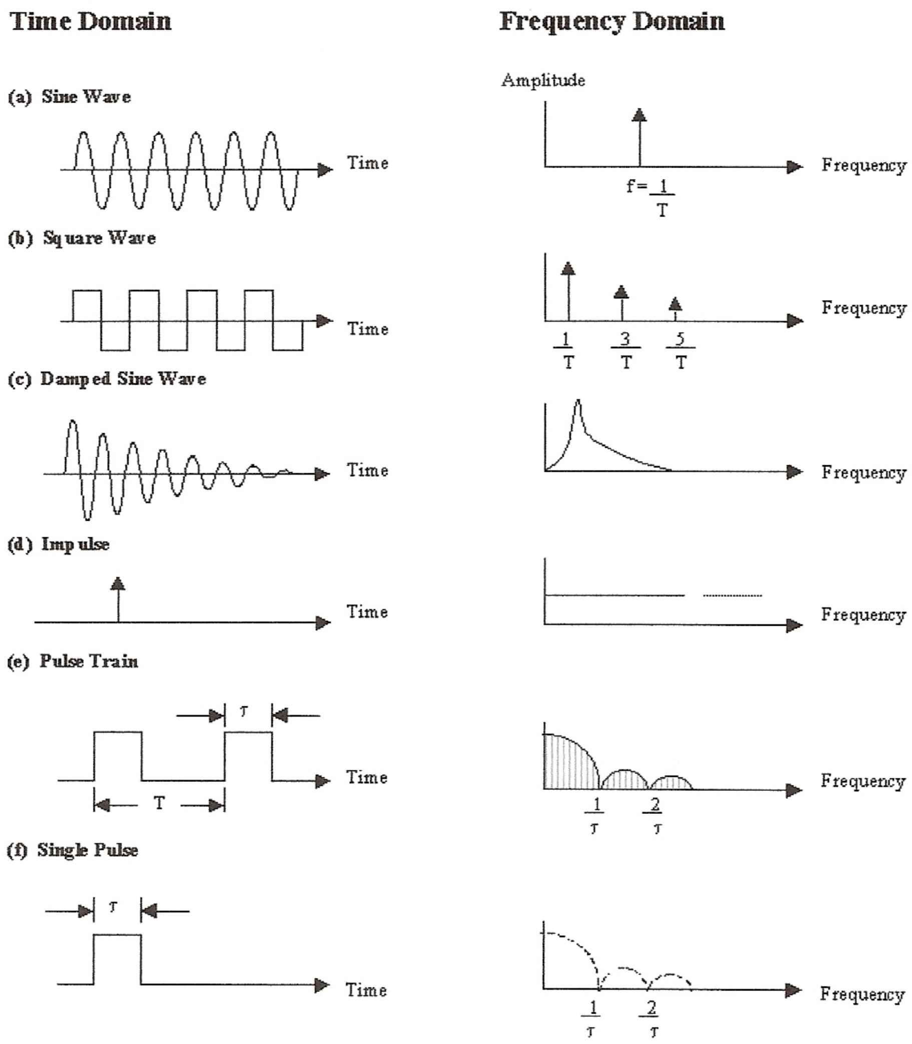

```
> p1:=plot(A1(w),w=-10..10,color=red):
  p2:=plot(A2(w),w=-10..10,color=blue):
> display(p1,p2);
```




```
>
```

$\Rightarrow$ We have seen that amplitude is $\dfrac{|F(\omega)|}{\pi}$ or $\dfrac{|\hat{f}(\xi)|}{\pi}$ and that periodic functions have discrete spectrum, while non-periodic ones have continuous spectrum.

Remark: examples of some spectra



Some properties of Fourier transform

1) **Linearity**, i.e.

$$\mathcal{F}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{F}\{f_1(t)\} + c_2 \mathcal{F}\{f_2(t)\}$$

c_1, c_2 are constants

2) **Shift in argument of original**

$$\mathcal{F}\{f(t-a)\} = e^{-i\omega a} \mathcal{F}\{f(t)\}$$

3) **Fourier transform of derivative**

we assume, that $f'(t)$ exists for $t \in (-\infty, \infty)$ and that $\int_{-\infty}^{\infty} |f'(t)|\, dt$ exists

(i.e. $\lim_{t \to -\infty} f(t) = 0$ and $\lim_{t \to +\infty} f(t) = 0$)

$$\underline{\mathcal{F}\{f'(t)\}} = \int_{-\infty}^{\infty} f'(t)\, e^{-i\omega t}\, dt =$$

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$$= \begin{bmatrix} u' = f' & v = e^{-i\omega t} \\ u = f & v' = -i\omega e^{-i\omega t} \end{bmatrix} = \underbrace{\left[f(t)\, e^{-i\omega t} \right]_{-\infty}^{\infty}}_{=0} +$$

$$+ \int_{-\infty}^{\infty} f(t)\, i\omega\, e^{-i\omega t}\, dt = \underline{i\omega\, \mathcal{F}\{f(t)\}}$$

Note there are no "initial conditions" with respect to Laplace transform.

$\rightarrow$ higher derivatives

$$\mathcal{F}\{f''\} = (i\omega)^2\, \mathcal{F}\{f\}$$

$$\vdots$$

$$\mathcal{F}\{f^{(k)}\} = (i\omega)^k\, \mathcal{F}\{f\}$$

4) <u>Fourier transform of convolution</u>

we know $\quad f(t) * g(t) = \int_{-\infty}^{\infty} f(s) \cdot g(t-s)\, ds =$

$$= \int_{-\infty}^{\infty} f(t-s)\, g(s)\, ds$$

(both integrals are possible since $f * g = g * f$)

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$$\underline{\underline{\mathcal{F}\{f*g\}}} = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(s)\, g(t-s)\, ds \right) e^{-i\omega t}\, dt =$$

$$= \left[\begin{array}{c} \text{substitution} \\ r = t-s \\ dr = dt \end{array} \right] = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(s)\, g(r)\, ds \right) e^{-i\omega (r+s)}\, dr =$$

$$= \int_{-\infty}^{\infty} f(s)\, e^{-i\omega s}\, ds \cdot \int_{-\infty}^{\infty} g(r)\, e^{-i\omega r}\, dr =$$

$$= \underline{\underline{\mathcal{F}\{f\} \cdot \mathcal{F}\{g\}}}$$

(the same property like for Laplace tr.)

Filtering of signal

Consider the signal $s(t) = \begin{cases} t, & 0 \le t \le 1 \\ 1, & 1 \le t \le 2 \\ 3-t, & 2 \le t \le 3 \end{cases}$

and $s(t)$ is zero outside $\langle 0;3 \rangle$.

We further consider noise

$n(t) = 0{,}1 \cdot \sin(20 \cdot \pi \cdot t)$ for $t \in \langle 1, 2 \rangle$.

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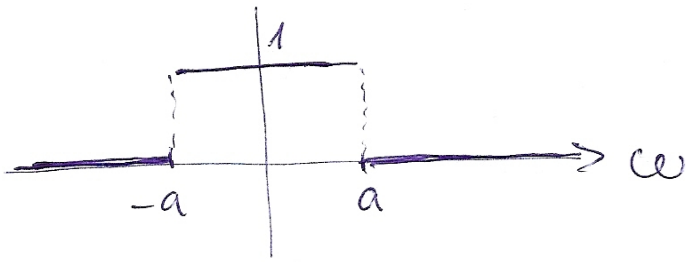
The "noisy" signal is $f(t) = s(t) + n(t)$

The idea is to perform Fourier transform of $f(t) \rightarrow F(\omega)$ and to apply filter $P_a(\omega)$ e.g. rect. filter



(low pass filter)

and filtered signal is then

$$f_{\text{filtered}}(t) = \mathcal{F}^{-1}\left\{F(\omega) \cdot P_a(\omega)\right\}$$

see the Maple file for examples with $P_a(\omega)$ for $a = 20\pi, 15\pi, 10\pi$ and 5π

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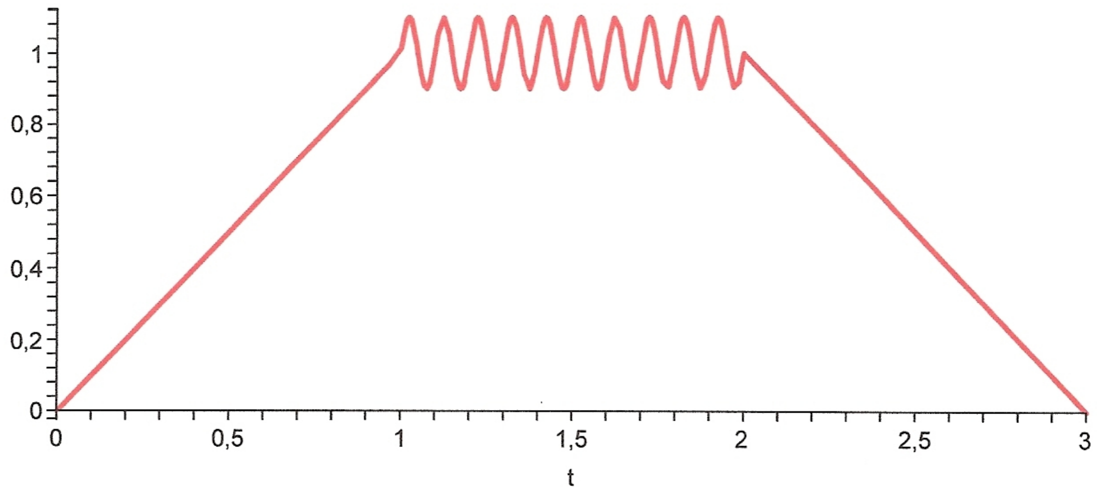
```
> restart:
> f(t):=piecewise(t>0 and t<1, t, t>1 and t<2, 1+0.1*sin(20*Pi*t), t>2 and
  t<3, 3-t);
```

$$f(t) := \begin{cases} t & 0 < t \text{ and } t < 1 \\ 1 + 0.1 \sin(20\,\pi\, t) & 1 < t \text{ and } t < 2 \\ 3 - t & 2 < t \text{ and } t < 3 \end{cases}$$

```
> with(plots):plot(f(t),t=0..3);
Warning, the name changecoords has been redefined
```




```
> F(w):=simplify(int(f(t)*exp(-I*t*w),t=0..3));
```

$$F(w) := -\frac{1}{w^2\,(6250.\,w^2 - 2.4674011\;10^7)}\,(0.00001000000000\,(6.25000000\;10^8\,w^2 - 2.467401100\;10^{12} + 3.301990817\;10^9\,e^{(-1.\,I\,w)}\,w^2 + 2.467401100\;10^{12}\,e^{(-1.\,I\,w)} - 4.551990817\;10^9\,e^{(-2.\,I\,w)}\,w^2 + 2.467401100\;10^{12}\,e^{(-2.\,I\,w)} + 6.25000000\;10^8\,e^{(-3.\,I\,w)}\,w^2 - 2.467401100\;10^{12}\,e^{(-3.\,I\,w)}))$$

```
> filtered1(t):=evalf(Int(F(w)*exp(I*t*w),w=-20*Pi..20*Pi)/2/Pi);
```
→ $a = 20\pi$

$$filtered1(t) := 0.1591549430 \int_{-62.83185308}^{62.83185308} -\frac{1}{w^2\,(6250.\,w^2 - 2.4674011\;10^7)}\,(0.00001000000000\,(6.25000000\;10^8\,w^2 - 2.467401100\;10^{12} + 3.301990817\;10^9\,e^{(-1.\,I\,w)}\,w^2 + 2.467401100\;10^{12}\,e^{(-1.\,I\,w)} - 4.551990817\;10^9\,e^{(-2.\,I\,w)}\,w^2 + 2.467401100\;10^{12}\,e^{(-2.\,I\,w)} + 6.25000000\;10^8\,e^{(-3.\,I\,w)}\,w^2 - 2.467401100\;10^{12}\,e^{(-3.\,I\,w)})\,e^{(1.\,I\,t\,w)})\,dw$$

```
> filtered2(t):=evalf(Int(F(w)*exp(I*t*w),w=-15*Pi..15*Pi)/2/Pi):
```
→ $a = 15\pi$

```
> filtered3(t):=evalf(Int(F(w)*exp(I*t*w),w=-10*Pi..10*Pi)/2/Pi):
```
→ $a = 10\pi$

> filtered 4(t) := ———— " ————, w=-5*Pi .. 5*Pi)/2/Pi): → $a = 5\pi$

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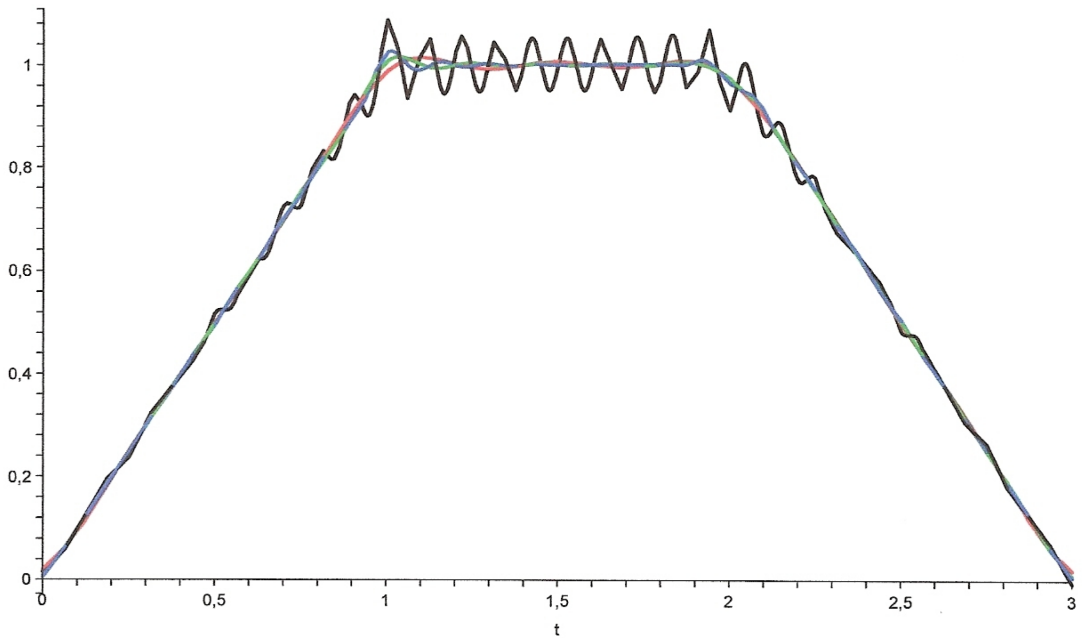
```
>
> p1:=plot(filtered1(t),t=0..3,color=black):
> p2:=plot(filtered2(t),t=0..3,color=blue):
> p3:=plot(filtered3(t),t=0..3,color=green):
> p4:=plot(filtered4(t),t=0..3,color=red):
> display(p1,p2,p3,p4);
```



Note we can also perform the inverse Fourier transform for filter $p_a(t) = \mathcal{F}^{-1}\{P_a(\omega)\}$

and then $f_{filtered}(t) = f(t) * p_a(t)$

$\left(\text{Since } F_{filtered}(\omega) = F(\omega) \cdot P_a(\omega) \right)$

See Maple file for $a = 5\pi$

```
>
>
```

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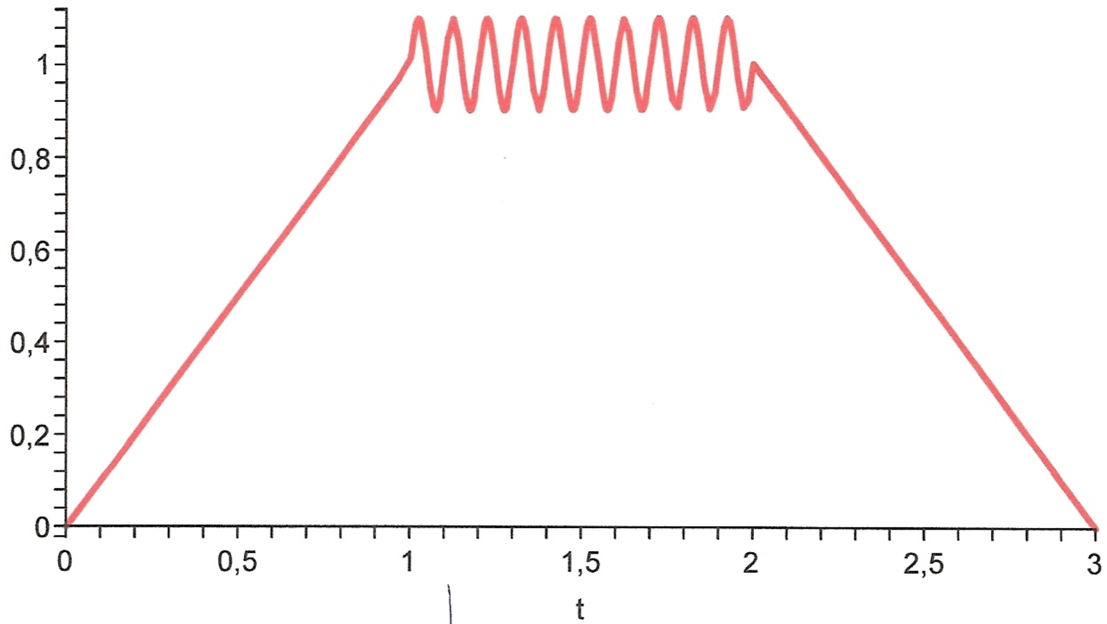
```
> restart:with(plots):
Warning, the name changecoords has been redefined
```

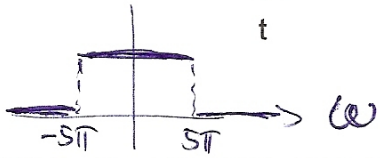
```
> f(t):=piecewise(t<0, 0, t>0 and t<1, t, t>1 and t<2, 1+0.1*sin(20*Pi*t),
  t>2 and t<3, 3-t, t>3, 0);
```

$$f(t) := \begin{cases} 0 & t < 0 \\ t & 0 < t \textbf{ and } t < 1 \\ 1 + 0.1\sin(20\pi t) & 1 < t \textbf{ and } t < 2 \\ 3 - t & 2 < t \textbf{ and } t < 3 \\ 0 & 3 < t \end{cases}$$

```
> plot(f(t),t=0..3);
```



filter




```
> G(w):=piecewise(w<-5*Pi, 0, w>-5*Pi and w<5*Pi, 1, w>5*Pi, 0);
```

$$G(w) := \begin{cases} 0 & w < -5\pi \\ 1 & -5\pi < w \textbf{ and } w < 5\pi \\ 0 & 5\pi < w \end{cases}$$

```
> with(inttrans):g(t):=invfourier(G(w),w,t);
```

$$g(t) := \frac{\sin(5\pi t)}{t\pi}$$

```
> plot(g(t),t=-2*Pi..2*Pi);
```

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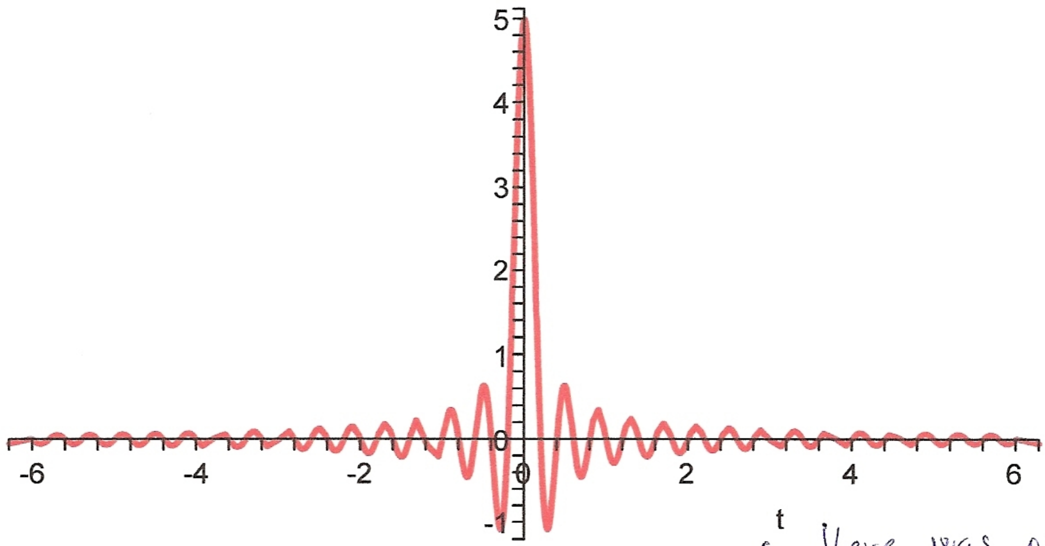
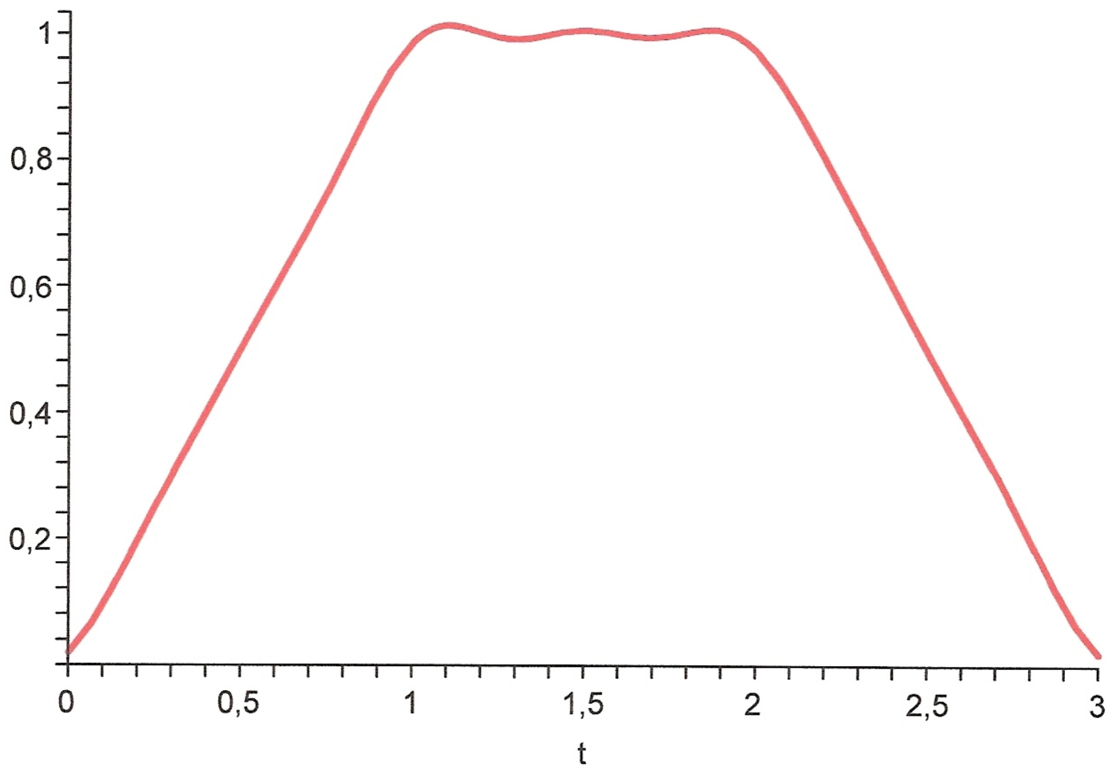


$\approx \int_{-\infty}^{\infty} f(t-s)g(s)\,ds$ (there was problem with $\int_{-\infty}^{\infty} f(s)g(t-s)ds$ due to numerics)

```
> convolution(t):=evalf(Int(subs(t=t-s,f(t))*subs(t=s,g(t)),s=-40*Pi..40*Pi)):
> plot(convolution(t),t=0..3);
```




```
>
```