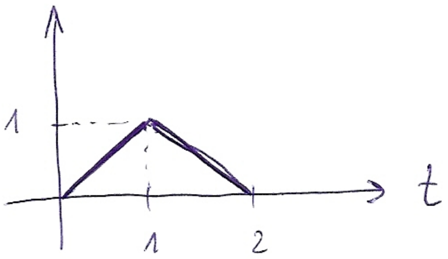
125

<u>Exercise:</u> Extend the given function

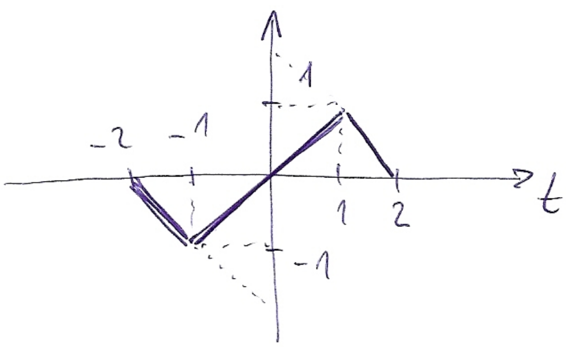


$t \in \langle 0; 2\rangle$ into

an odd periodic function and find the Fourier series for this extension.

---

the odd extension is (period = 4)



$$f(t) = \begin{cases} -t-2, & t \in \langle -2; -1) \\ t, & t \in \langle -1; 1) \\ -t+2, & t \in \langle 1; 2) \end{cases}$$

odd function $\Rightarrow$ $a_k = 0$, $k = 0, 1, \dots$

i.e. $f(t) = \sum_{k=1}^{\infty} \left[b_k \sin\left(\frac{2\pi k t}{4}\right) \right]$

with $b_k = \frac{2}{4} \int_{-2}^{2} \underbrace{\underbrace{f(t)}_{\text{odd}} \underbrace{\sin\left(\frac{2\pi k t}{4}\right)}_{\text{odd}}}_{\text{even}} dt =$

$= \frac{2}{4} \cdot 2 \int_{0}^{2} f(t) \sin\left(\frac{\pi k t}{2}\right) dt$

(126)

```
> f:=piecewise(t>-2 and t<-1,-t-2,t>-1 and t<1,t,t>1 and t<2,-t+2);
```

$$f := \begin{cases} -t-2 & -2 < t \text{ and } t < -1 \\ t & -1 < t \text{ and } t < 1 \\ -t+2 & 1 < t \text{ and } t < 2 \end{cases}$$

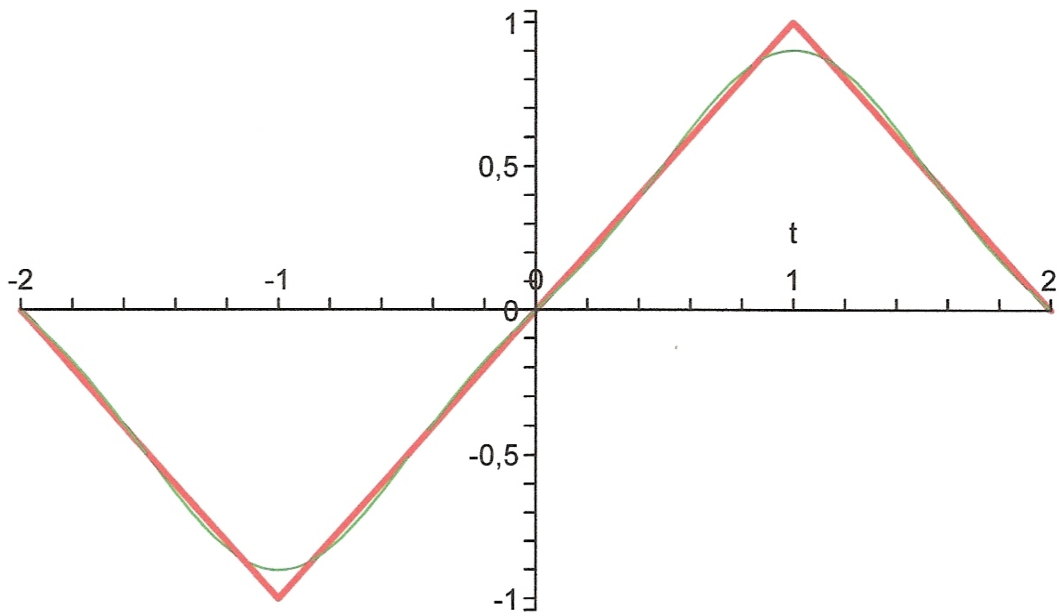
```
> bk:=int(f*sin(Pi*k*t/2),t=0..2);
```

$$bk := -\frac{2\left(-2\sin\left(\frac{\pi k}{2}\right)+\cos\left(\frac{\pi k}{2}\right)\pi k\right)}{\pi^2 k^2} + \frac{2\left(2\sin\left(\frac{\pi k}{2}\right)+\cos\left(\frac{\pi k}{2}\right)\pi k - 2\sin(\pi k)\right)}{\pi^2 k^2}$$

```
> s:=sum(bk*sin(Pi*k*t/2),k=1..4);
```

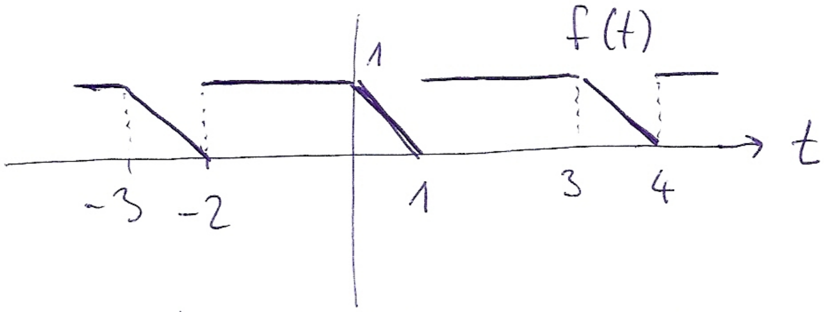
$$s := \frac{8\sin\left(\frac{\pi t}{2}\right)}{\pi^2} - \frac{8}{9}\,\frac{\sin\left(\frac{3\pi t}{2}\right)}{\pi^2}$$

```
> with(plots):
> plot([f,s],t=-2..2);
```




```
>
```

Exercise: Find the Fourier series for periodic function $f(t)$ with period $= 3$.



$f(t)$ is neither odd nor even $\Rightarrow$

$$\Rightarrow \quad f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{2\pi k t}{3}\right) + b_k \sin\left(\frac{2\pi k t}{3}\right) \right)$$

where
$$a_k = \frac{2}{3} \int_{-2}^{1} f(t) \cos\left(\frac{2\pi k t}{3}\right) dt =$$

$$= \frac{2}{3} \int_{-2}^{0} 1 \cdot \cos\left(\frac{2\pi k t}{3}\right) dt + \frac{2}{3} \int_{0}^{1} (1-t) \cdot \cos\left(\frac{2\pi k t}{3}\right) \cdot dt = \text{----}$$

$$b_k = \frac{2}{3} \int_{-2}^{1} f(t) \sin\left(\frac{2\pi k t}{3}\right) dt = \text{-----}$$

$$a_0 = \frac{2}{3} \int_{-2}^{1} f(t)\, dt = \frac{2}{3} \cdot \text{"surface"} =$$

$$= \frac{\cancel{2}}{3} \cdot \frac{5}{\cancel{2}} = \frac{5}{3}$$

for further result see Maple file (note there is "x" instead of "t")

128

```
> f:=piecewise(x<-3,1,x>-3 and x<-2,-x-2,x>-2 and x<0,1, x>0 and x<1,1-x,x>1 and x<3,1,x>3 and x<4,4-x);
```

$$f := \begin{cases} 1 & x < -3 \\ -x-2 & -3 < x \textbf{ and } x < -2 \\ 1 & -2 < x \textbf{ and } x < 0 \\ 1-x & 0 < x \textbf{ and } x < 1 \\ 1 & 1 < x \textbf{ and } x < 3 \\ 4-x & 3 < x \textbf{ and } x < 4 \end{cases}$$

```
> a0:=2/3*int(f,x=-2..1);
```

$$a0 := \frac{5}{3}$$

```
> ak:=2/3*int(f*cos(2*Pi*k*x/3),x=-2..1);
```

$$ak := \frac{\sin\left(\frac{4\pi k}{3}\right)}{\pi k} - \frac{3}{2}\,\frac{-1+\cos\left(\frac{2\pi k}{3}\right)}{\pi^2 k^2}$$

```
> bk:=2/3*int(f*sin(2*Pi*k*x/3),x=-2..1);
```

$$bk := \frac{\cos\left(\frac{4\pi k}{3}\right) - 1}{\pi k} + \frac{1}{2}\,\frac{2\pi k - 3\sin\left(\frac{2\pi k}{3}\right)}{\pi^2 k^2}$$

```
> s:=a0/2+sum(ak*cos(2*Pi*k*x/3)+bk*sin(2*Pi*k*x/3),k=1..8);
```

$$s := \frac{5}{6} + \left(-\frac{\sqrt{3}}{2\pi} + \frac{9}{4\pi^2}\right)\cos\left(\frac{2\pi x}{3}\right) + \left(-\frac{3}{2\pi} + \frac{2\pi - \frac{3\sqrt{3}}{2}}{2\pi^2}\right)\sin\left(\frac{2\pi x}{3}\right)$$

$$+ \left(\frac{\sqrt{3}}{4\pi} + \frac{9}{16\pi^2}\right)\cos\left(\frac{4\pi x}{3}\right) + \left(-\frac{3}{4\pi} + \frac{4\pi + \frac{3\sqrt{3}}{2}}{8\pi^2}\right)\sin\left(\frac{4\pi x}{3}\right) + \frac{1}{3}\,\frac{\sin(2\pi x)}{\pi}$$

$$+ \left(-\frac{\sqrt{3}}{8\pi} + \frac{9}{64\pi^2}\right)\cos\left(\frac{8\pi x}{3}\right) + \left(-\frac{3}{8\pi} + \frac{8\pi - \frac{3\sqrt{3}}{2}}{32\pi^2}\right)\sin\left(\frac{8\pi x}{3}\right)$$

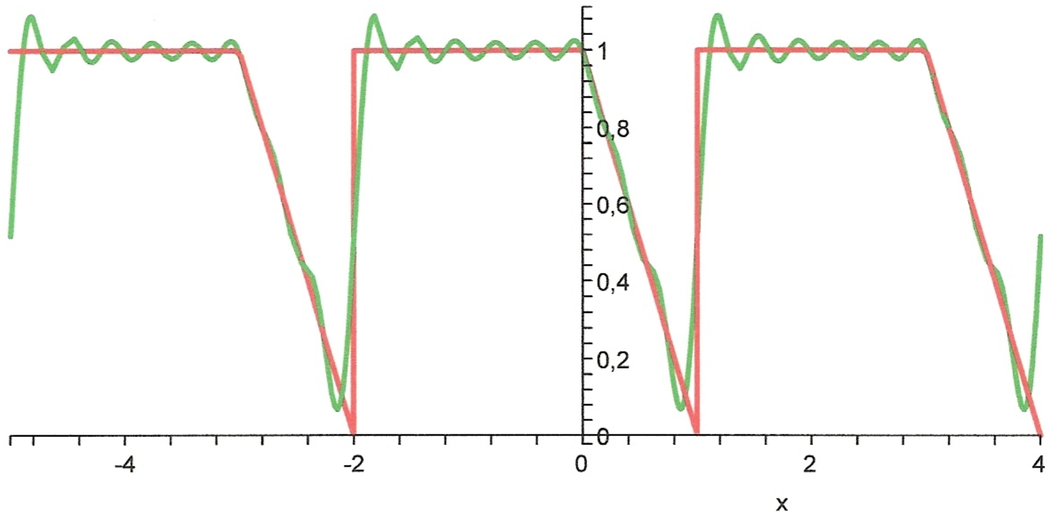
$$+ \left(\frac{\sqrt{3}}{10\pi} + \frac{9}{100\pi^2}\right)\cos\left(\frac{10\pi x}{3}\right) + \left(-\frac{3}{10\pi} + \frac{10\pi + \frac{3\sqrt{3}}{2}}{50\pi^2}\right)\sin\left(\frac{10\pi x}{3}\right) + \frac{1}{6}\,\frac{\sin(4\pi x)}{\pi}$$

$$+ \left(-\frac{\sqrt{3}}{14\pi} + \frac{9}{196\pi^2}\right)\cos\left(\frac{14\pi x}{3}\right) + \left(-\frac{3}{14\pi} + \frac{14\pi - \frac{3\sqrt{3}}{2}}{98\pi^2}\right)\sin\left(\frac{14\pi x}{3}\right)$$

$$+ \left(\frac{\sqrt{3}}{16\pi} + \frac{9}{[illegible]}\right)\cos\left(\frac{16\pi x}{[illegible]}\right) + \left(-\frac{3}{[illegible]} + \frac{16\pi + \frac{3\sqrt{3}}{2}}{[illegible]}\right)\sin\left(\frac{16\pi x}{[illegible]}\right)$$

129

```
> with(plots):
> plot([f,s],x=-5..4);
```




```
>
```

Note the behavior of sum of Fourier series in discontinuity, where it converges to the "average"

$$\left(\frac{\lim\limits_{t \to t_{0-}} f(t) + \lim\limits_{t \to t_{0+}} f(t)}{2} \right)$$

(130)

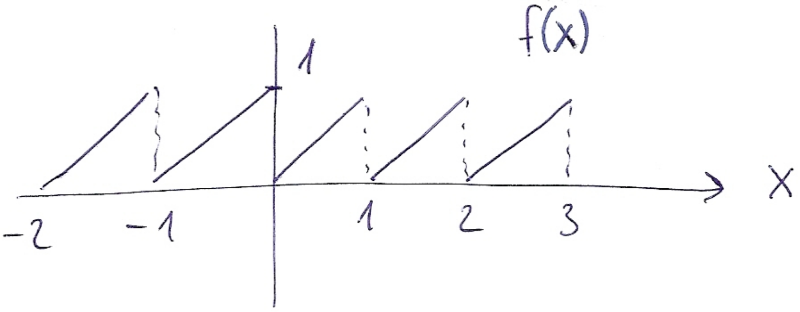
<u>Exercise:</u> Approximate the puls (function)

$f(x) = x, \ x \in \langle 0;1 \rangle$ by the sum of

Fourier series.

We need periodic function, i.e. we have to extend the given function to periodic function. Let's try different extensions:

1) "simple extension"



period $= 1$

$f(x)$ is neither odd nor even

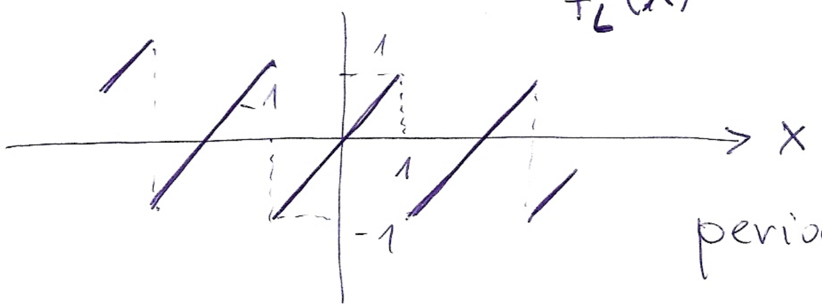
$$f(x) \doteq S(x) = \frac{a_0}{2} + \sum_{k=1}^{N} \left(a_k \cos\left(\frac{2\pi k x}{1}\right) + b_k \sin\left(\frac{2\pi k x}{1}\right)\right)$$

$$a_k = \frac{2}{1} \int_0^1 x \cos\left(\frac{2\pi k x}{1}\right) dx = \dots$$

$b_k = \dots$

131

2) "odd extension"



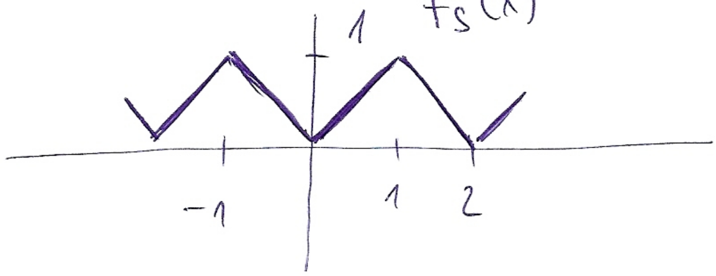
period $=2$

$a_k = 0$

$$f_L(x) \doteq S_L(x) = \sum_{k=1}^{N} \left[b_k \sin\left(\frac{2\pi k x}{2}\right) \right]$$

$$b_k = \frac{2}{2} \int_{-1}^{1} x \sin\left(\frac{2\pi k x}{2}\right) dx = \cdots$$

3) "even extension"



period $=2$

$b_k = 0$

$$f_S(x) \doteq S_S(x) = \frac{a_0}{2} + \sum_{k=1}^{N} \left[a_k \cos\left(\frac{2\pi k x}{2}\right) \right]$$

$$a_k = \frac{2}{2}\, 2 \int_{0}^{1} x \cos\left(\frac{2\pi k x}{2}\right) dx = \cdots$$

```
> restart:
```

```
> f:=piecewise(x<0,0,x>0 and x<1,x,x>1,0);
```

$$f := \begin{cases} 0 & x < 0 \\ x & 0 < x \textbf{ and } x < 1 \\ 0 & 1 < x \end{cases}$$

```
> a0:=2/1*int(f,x=0..1);
```

$$a0 := 1$$

```
> ak:=2/1*int(f*cos(2*Pi*k*x/1),x=0..1);
```

$$ak := \frac{-1 + \cos(\pi k)^2 + 2 \pi k \sin(\pi k) \cos(\pi k)}{\pi^2 k^2}$$

```
> bk:=2/1*int(f*sin(2*Pi*k*x/1),x=0..1);
```

$$bk := -\frac{-\sin(\pi k) \cos(\pi k) + 2 \pi k \cos(\pi k)^2 - \pi k}{\pi^2 k^2}$$

```
> s:=a0/2+sum(ak*cos(2*Pi*k*x/1)+bk*sin(2*Pi*k*x/1),k=1..8);
```

$$s := \frac{1}{2} - \frac{\sin(2 \pi x)}{\pi} - \frac{\sin(4 \pi x)}{2 \pi} - \frac{\sin(6 \pi x)}{3 \pi} - \frac{\sin(8 \pi x)}{4 \pi} - \frac{\sin(10 \pi x)}{5 \pi} - \frac{\sin(12 \pi x)}{6 \pi} - \frac{\sin(14 \pi x)}{7 \pi} - \frac{\sin(16 \pi x)}{8 \pi}$$

```
> fL:=piecewise(x<-1,0,x>-1 and x<1,x,x>1,0);
```

$$fL := \begin{cases} 0 & x < -1 \\ x & -1 < x \textbf{ and } x < 1 \\ 0 & 1 < x \end{cases}$$

```
> bLk:=2/2*int(fL*sin(2*Pi*k*x/2),x=-1..1);
```

$$bLk := -\frac{2(-\sin(\pi k) + \cos(\pi k) \pi k)}{\pi^2 k^2}$$

```
> sL:=sum(bLk*sin(2*Pi*k*x/2),k=1..8);
```

$$sL := \frac{2 \sin(\pi x)}{\pi} - \frac{\sin(2 \pi x)}{\pi} + \frac{2 \sin(3 \pi x)}{3 \pi} - \frac{\sin(4 \pi x)}{2 \pi} + \frac{2 \sin(5 \pi x)}{5 \pi} - \frac{\sin(6 \pi x)}{3 \pi} + \frac{2 \sin(7 \pi x)}{7 \pi} - \frac{\sin(8 \pi x)}{4 \pi}$$

```
> fS:=piecewise(x<-1,0,x>-1 and x<0,-x,x>0 and x<1,x,x>1,0);
```

$$fS := \begin{cases} 0 & x < -1 \\ -x & -1 < x \textbf{ and } x < 0 \\ x & 0 < x \textbf{ and } x < 1 \\ 0 & 1 < x \end{cases}$$

```
> aS0:=2/2*int(fS,x=-1..1);
```

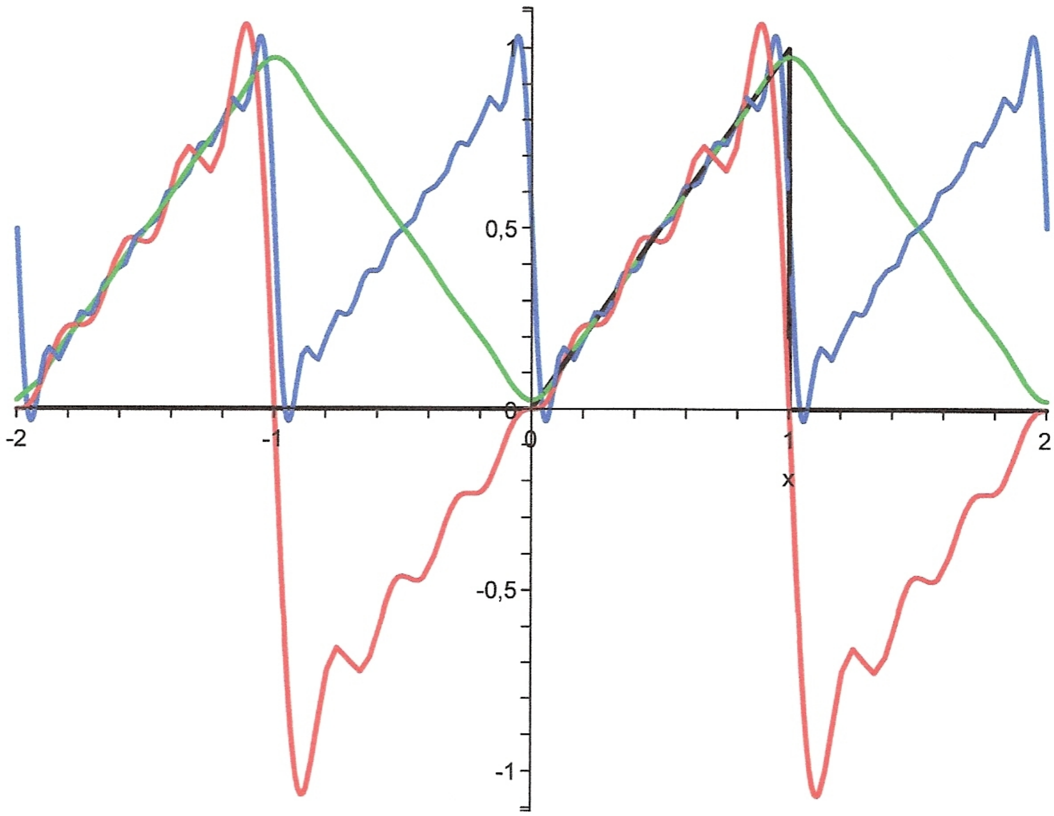
$$aS0 := 1$$

```
> aSk:=2/2*int(fS*cos(2*Pi*k*x/2),x=-1..1);
```

$$aSk := \frac{2(\cos(\pi k) + \sin(\pi k) \pi k - 1)}{\pi^2 k^2}$$

```
> sS:=aS0/2+sum(aSk*cos(2*Pi*k*x/2),k=1..8);
```

```
> with(plots):
> p1:=plot(f,x=-2..2,color=black):
> p2:=plot(s,x=-2..2,color=blue):
> p3:=plot(sL,x=-2..2,color=red):
> p4:=plot(sS,x=-2..2,color=green):
> display(p1,p2,p3,p4);
```



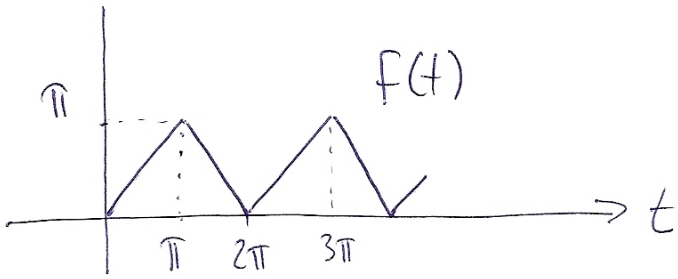
Rem: The "best" approximation of the given puls is the sum of Fourier series for even extension, which is continuous and therefore it does not suffer from oscillations.

```
>
```

(134)

Exercise: Find the solution of

$y''(t) + y'(t) + y(t) = f(t)$ corresponding to forcing function $f(t)$, which is periodic with period 2π, see fig.



The solution of $y''(t) + y'(t) + y(t) = f(t)$ has the form $y(t) = y_H(t) + y_P(t)$,

where $y_H(t)$ is the homogenous part (or transient)

and $\underline{\underline{y_P(t)}}$ is the response to $f(t)$

"our goal"

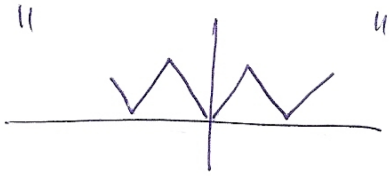
→ homogenous part

$\lambda^2 + \lambda + 1 = 0 \;\rightarrow\; \lambda_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\rightarrow \; \underset{\sim\sim\sim}{y_H(t) = e^{-t/2} \left(c_1 \cos\left(\frac{\sqrt{3}}{2} t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2} t\right) \right)}$

→ response to $f(t)$:

we can estimate $y_p(t)$ if $f(t)$ is polynomial, exponential or trigonometric function ⇒ let's write $f(t)$ as the sum of Fourier series

$f(t)$ is even "  " ⇒ $b_n = 0$

$$f(t) = \frac{a_0}{2} - \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{2\pi k t}{2\pi}\right) \right]$$

where

$$a_k = \frac{\cancel{2}}{\cancel{2}\pi} \int_{-\pi}^{\pi} f(t) \cos(kt)\, dt =$$

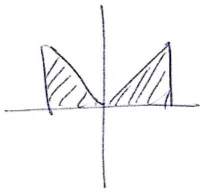
$$= \frac{1}{\pi}\, 2 \int_0^{\pi} t \cos(kt)\, dt = \begin{bmatrix} u' = \cos(kt) & v = t \\ u = \frac{\sin(kt)}{k} & v' = 1 \end{bmatrix} =$$

$$= \underbrace{\frac{2}{\pi} \left[t\, \frac{\sin(kt)}{k} \right]_0^{\pi}}_{=0} - \frac{2}{\pi} \frac{1}{k} \int_0^{\pi} \sin(kt)\, dt =$$

$$= -\frac{2}{\pi k} \left[-\frac{\cos(kt)}{k} \right]_0^{\pi} = -\frac{2}{\pi k^2}\left(-\cos(k\pi) + 1\right) =$$

136

$$= -\frac{2}{\pi k^2}\left(-(-1)^k+1\right) = \begin{cases} -\dfrac{4}{\pi k^2}, & \underbrace{k=1,3,5,\dots}_{2j+1,\ j=0,1,\dots} \\ 0, & k=2,4,6,\dots \end{cases}$$

$a_0 = \frac{2}{2\pi}\cdot$ "surface of  " $= \frac{2}{2\pi}\cdot\pi^2 =$

$= \pi$

$$\Rightarrow \underbrace{f(t) = \frac{\pi}{2} + \sum_{j=0}^{\infty} -\frac{4}{\pi(2j+1)^2}\cos\left[(2j+1)t\right]}_{\Downarrow \text{ estimate for } y_P(t)} \quad (1)$$

different from $\frac{\sqrt{3}}{2}$ in y_H

$$y_P(t) = \tilde{y}_0 + \sum_{j=0}^{\infty}\left(y_j \cos\left[(2j+1)t\right] + z_j \sin\left[(2j+1)t\right]\right) \quad (2)$$

let's substitute (1) and (2) into

$y_P'' + y_P' + y_P = f$ and find

unknown coefficients $\tilde{y}_0$, y_j, z_j, $j=0,1,\dots$

137

$$y_p'(t) = \sum_{j=0}^{\infty} \left(-y_j (2j+1) \sin[-''-] + z_j (2j+1) \cos[-''-] \right)$$

$$y''_p(t) = \sum_{j=0}^{\infty} \left(-y_j (2j+1)^2 \cos[-''-] - z_j (2j+1)^2 \sin[-''-] \right)$$

$$\tilde{y}_0 = \frac{\pi}{2}$$

$$\left.\begin{aligned} y_j + z_j (2j+1) - y_j (2j+1)^2 &= -\frac{4}{\pi(2j+1)^2} \\ z_j - y_j (2j+1) - z_j (2j+1)^2 &= 0 \end{aligned}\right\} \quad j = 0, 1, \dots$$

---

for $y_j = \dots$, $z_j = \dots$

see Maple file

138

```
> restart:
> y0:=Pi;
```

$$y0 := \pi$$

```
> koef:=solve({(1-(2*j+1)^2)*yj+(2*j+1)*zj=-4/Pi/(2*j+1)^2,
  -(2*j+1)*yj+(1-(2*j+1)^2)*zj=0}, {yj,zj});
```

$$koef := \left\{ yj = \frac{16\,j\,(j+1)}{(2\,j+1)\,(32\,j^5+80\,j^4+72\,j^3+28\,j^2+6\,j+1)\,\pi},\; zj = -\frac{4}{(32\,j^5+80\,j^4+72\,j^3+28\,j^2+6\,j+1)\,\pi} \right\}$$

```
> assign(koef);
> yj;
```

$$\frac{16\,j\,(j+1)}{(2\,j+1)\,(32\,j^5+80\,j^4+72\,j^3+28\,j^2+6\,j+1)\,\pi}$$

```
> zj;
```

$$-\frac{4}{(32\,j^5+80\,j^4+72\,j^3+28\,j^2+6\,j+1)\,\pi}$$

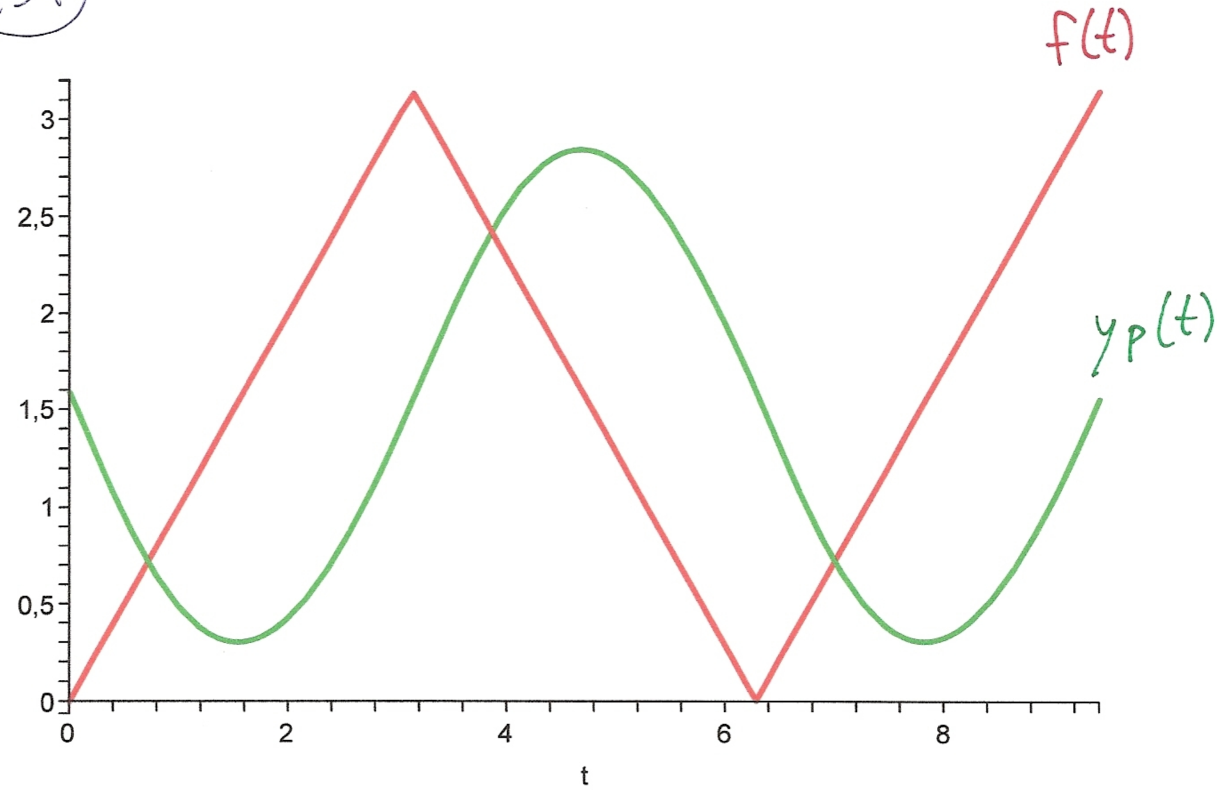
```
> y(t):=y0/2+sum(yj*cos((2*j+1)*t)+zj*sin((2*j+1)*t),j=0..8);
```

$$y(t) := \frac{\pi}{2} - \frac{4\sin(t)}{\pi} + \frac{32}{657}\frac{\cos(3\,t)}{\pi} - \frac{4}{219}\frac{\sin(3\,t)}{\pi} + \frac{96}{15025}\frac{\cos(5\,t)}{\pi} - \frac{4}{3005}\frac{\sin(5\,t)}{\pi} + \frac{192}{115297}\frac{\cos(7\,t)}{\pi}$$
$$- \frac{4}{16471}\frac{\sin(7\,t)}{\pi} + \frac{320}{524961}\frac{\cos(9\,t)}{\pi} - \frac{4}{58329}\frac{\sin(9\,t)}{\pi} + \frac{480}{1757041}\frac{\cos(11\,t)}{\pi} - \frac{4}{159731}\frac{\sin(11\,t)}{\pi}$$
$$+ \frac{672}{4798417}\frac{\cos(13\,t)}{\pi} - \frac{4}{369109}\frac{\sin(13\,t)}{\pi} + \frac{896}{11340225}\frac{\cos(15\,t)}{\pi} - \frac{4}{756015}\frac{\sin(15\,t)}{\pi}$$
$$+ \frac{1152}{24054337}\frac{\cos(17\,t)}{\pi} - \frac{4}{1414961}\frac{\sin(17\,t)}{\pi}$$

```
> f(t):=piecewise(t>0 and t<Pi, t, t>Pi and t<2*Pi, 2*Pi-t, t>2*Pi and
  t<3*Pi, t-2*Pi);
```

$$f(t) := \begin{cases} t & 0 < t \textbf{ and } t < \pi \\ 2\,\pi - t & \pi < t \textbf{ and } t < 2\,\pi \\ t - 2\,\pi & 2\,\pi < t \textbf{ and } t < 3\,\pi \end{cases}$$

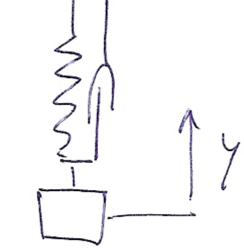
```
> with(plots):
Warning, the name changecoords has been redefined
> plot([f(t),y(t)],t=0..3*Pi);
```



Remark: physical interpretation

$$1\cdot y'' + 1\cdot y' + 1\cdot y = f$$

mass friction spring force



$y_p(t)$ ... response to force $f(t)$ without transient (y_H)

Application of Fourier series to the solution of PDE's

We consider following problem for parabolic equation:

$$\frac{\partial v(x,t)}{\partial t} = \alpha \frac{\partial^2 v(x,t)}{\partial x^2}, \quad \alpha > 0 \qquad (1)$$

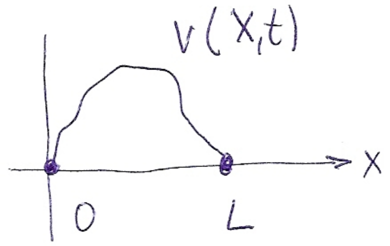
$$[x,t] \in (0,L) \times (0,\infty)$$

initial condition: $v(x,0) = \underbrace{V_0(x)}_{\text{given function}}, \quad x \in \langle 0, L \rangle$

boundary conditions: $\left.\begin{aligned} v(0,t) &= 0 \\ v(L,t) &= 0 \end{aligned}\right\} \; t \geq 0$

We know that solution $v(x,t)$ at any time t is some function of x, which is equal to zero for $x=0$ and $x=L$ and is defined for $x \in \langle 0, L \rangle$.

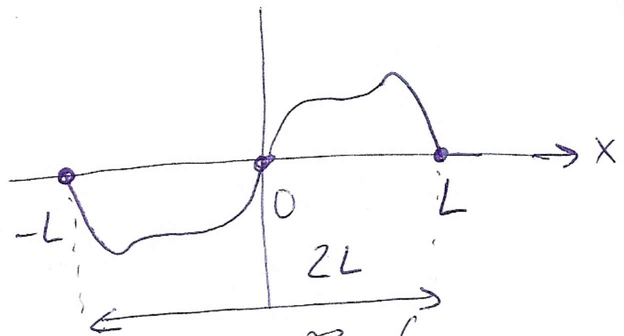
We can extend the solution $v(x,t)$ for any time t to periodic function and to take it as a sum of Fourier series.



which periodic extension ?

→ we prefer <u>continuous</u> periodic extension (to avoid oscillation at discontinuities)

The best periodic for our case is the odd extension with period $2L$



$$v(x,t) = \sum_{k=1}^{\infty}\left(b_k(t)\,\sin\frac{2\pi k x}{2L}\right) \qquad (2)$$

since all functions $\sin\frac{2\pi k x}{2L}$ satisfy automatically the boundary conditions ($\sin 0 = 0$, $\sin\frac{2\pi k L}{2L} = 0$). Note that coefficients $b_k(t)$ are now the functions of t.

142

Let's substitute (2) into (1)

$$\frac{\partial v}{\partial t} = \sum_{k=1}^{\infty} b_k'(t) \sin\frac{k\pi x}{L}$$

$$\frac{\partial^2 v}{\partial x^2} = \sum_{k=1}^{\infty} -b_k(t)\left(\frac{k\pi}{L}\right)^2 \sin\frac{k\pi x}{L}$$

$$b_k'(t) = -\alpha\, b_k(t)\left(\frac{k\pi}{L}\right)^2, \quad k = 1, 2, \dots, \infty$$

$$\Rightarrow b_k(t) = C_K \cdot \exp\left(-\alpha \frac{k^2\pi^2}{L^2} t\right)$$

$\Rightarrow$ the solution is

$$v(x,t) = \sum_{k=1}^{\infty} C_K \exp\left(-\alpha \frac{k^2\pi^2}{L^2} t\right) \sin\frac{k\pi x}{L}$$

?... these coefficients have to be set according to initial condition

we know $$v(x,0) = \sum_{k=1}^{\infty} C_K \sin\left(\frac{2\pi k x}{2L}\right) = V_0(x)$$

given initial cond.

$\Rightarrow$ we have to compute Fourier coefficients for odd extension of $V_0(x)$ with period $2L$

$$\Rightarrow \quad C_K = \frac{\cancel{2}}{\cancel{2}L} \int_{-L}^{L} \text{"extension of } v_0(x)\text{"} \cdot \sin\frac{\cancel{2}\pi k x}{\cancel{2}L}\, dx =$$

$$= \frac{1}{L}\, 2 \int_0^L V_0(x) \sin\frac{\pi k x}{L}\, dx$$

Remark: The above described method is refered as Fourier method or separation of variables method.

Remark: If we consider Neumann boundary conditions i.e. problem

$$\frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial x^2}$$

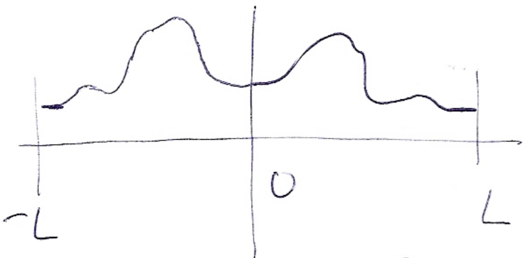
$$v(x,0) = v_0(x), \; x \in \langle 0, L \rangle$$

$$\frac{\partial v}{\partial x}(0,t) = \frac{\partial v}{\partial x}(L,t) = 0, \; t \geq 0$$

then we estimate the solution in form

$$v(x,t) = a_0(t) + \sum_{k=1}^{\infty} a_k(t) \cos\left(\frac{2k\pi x}{2L}\right)$$

i.e. we use even extension of $v(x,t)$ into periodic function with period $2L$



, which is continuous and since all functions $\cos\left(\frac{2k\pi x}{2L}\right)$ satisfy automatically the boundary conditions.

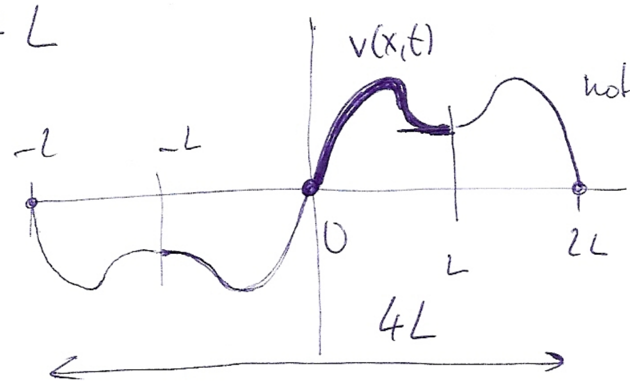
Remark: in the case of boundary conditions

$$v(0,t) = 0 \quad \text{and} \quad \frac{\partial v}{\partial x}(L,t) = 0 \quad \text{for } t \geq 0$$

we use the estimate of solution

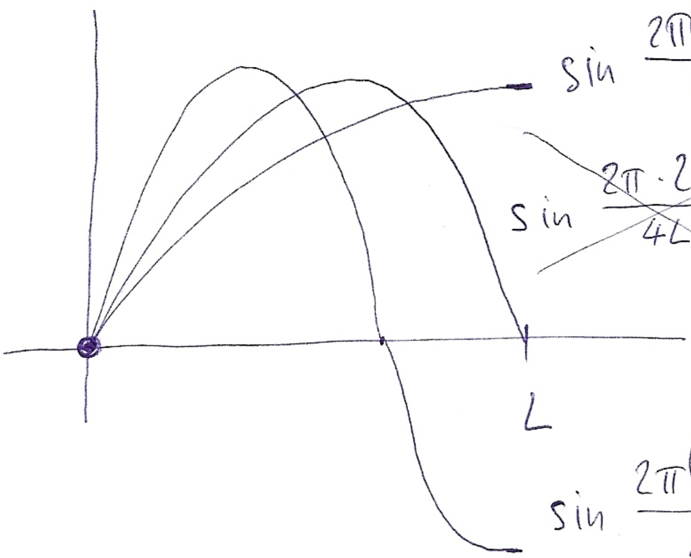
$$v(x,t) = \sum_{j=0}^{\infty} b_j(t) \sin \frac{2\pi(2j+1)x}{4L}$$

i.e. the periodic extension of $v(x,t)$ with period $4L$



note it is continuous function again

and functions $\sin \frac{2\pi(2j+1)x}{4L}$ satisfy again automatically boundary conditions



$\sin \frac{2\pi \cdot 1 \cdot x}{4L}$

$\cancel{\sin \frac{2\pi \cdot 2 \cdot x}{4L}}$

$\sin \frac{2\pi \cdot 3 \cdot x}{4L}$

we need only odd numbers because of $\frac{\partial v}{\partial x}(L,t) = 0$