We again need $|b| < 1$, i.e.

$$\left| \frac{RC - h}{RC} \right| < 1 \quad \Rightarrow \quad h \in (0; 2RC)$$

So the realization (B) of RC filter is stable only for $h \in (0; 2RC)$.

FOURIER SERIES

We assume some periodic real function $f(t)$ with period $a > 0$, i.e. $f(t+a) = f(t)$ for any $t \in \mathbb{R}$.

Definition: we define the trigonometric polynomial of degree N as the function

$$p(t) = \sum_{k=-N}^{N} c_k e^{2\pi i k \frac{t}{a}},$$

where c_k is complex, $a \in \mathbb{R}$, $a > 0$, $N > 0$ and i is complex unit $(\sqrt{-1})$.

Remark: If we apply the formula
$e^{ix} = \cos x + i \sin x$, we can write
the trigonometric polynomial in another
form:

$$p(t) = c_0 + \sum_{k=1}^{N} \left(c_k e^{2\pi i k \frac{t}{a}} + c_{-k} e^{-2\pi i k \frac{t}{a}} \right) =$$

$$= \frac{a_0}{2} + \sum_{k=1}^{N} \left(a_k \cos\left(\frac{2\pi k t}{a}\right) + b_k \sin\left(\frac{2\pi k t}{a}\right)\right),$$

where $a_k = c_k + c_{-k}$, $k = 0, \dots, N$

$b_k = i(c_k - c_{-k})$, $k = 1, \dots, N$

Now let's make a short trip to functional analysis. We consider the set (in czech množinu) of all trigonometric polynomials of degree N. We define the scalar product of two polynomials p and q as

$$(p, q) = \int_0^a p(t) \cdot \bar{q}(t)\, dt .$$

We further define the norm of polynomial p induced by scalar product as

$$\|p\|_2 = \sqrt{(p,p)} = \sqrt{\int_0^a |p|^2 dt} \,. \qquad (N)$$

The set of all trigonometric polynomials of degree N together with norm (N) define the $(2N+1)$ - dimensional vectorial space T_N (note that the dimension of space $2N+1$ is given by the number of independent coefficients in definition of $p(t)$, i.e. c_k, $k=-N, \dots, N$).

Let's denote $\quad e_k(t) = e^{2\pi i k \frac{t}{a}}$.

Remark: important properties of $e_k(t)$:

it is possible to prove

$$\int_0^a e_n(t)\, \overline{e_m}(t)\, dt = \begin{cases} a, & m=n \\ 0, & m \neq n \end{cases}$$

i.e. $$(e_n, e_m) = 0 \quad \text{for } n \neq m$$
$$(e_n, e_n) = a$$
$$\| e_n \|_2 = \sqrt{a}$$

We can show, that any element of T_N can be written as a linear combination of $e_k(t)$, i.e.

$$p(t) = \sum_{k=-N}^{N} c_k e_k(t)$$

Remark: Note the similarity with algebraic vectors. We also say, that functions $e_k(t)$, $k = -N, \dots, N$ are the base of T_N and due to property $(e_n, e_m) = 0$, $m \neq n$ we speak about the orthogonal base of T_N.

Now consider $p(t) \in T_N$ is given and we want to compute $c_k = ?$

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Let's write the scalar product of equation

$$p(t) = \sum_{k=-N}^{N} c_k e_k(t)$$

with the function $e_n(t)$:

$$\underline{\underline{(p(t), e_n(t))}} = \left(\sum_{k=-N}^{N} c_k e_k(t), e_n \right) =$$

$$= \sum_{k=-N}^{N} c_k (e_k(t), e_n(t)) = c_n (e_n(t),$$

$$e_n(t)) = \underline{\underline{c_n \cdot a}}$$

$$\Rightarrow \underline{\underline{c_n}} = \frac{1}{a}(p, e_n) = \underline{\underline{\frac{1}{a}\int_0^a p(t) e^{-2\pi i n \frac{t}{a}} dt}}$$

Remark: if we consider the formula

$$p(t) = \frac{a_0}{2} + \sum_{k=1}^{N} \left[a_k \cos\left(\frac{2\pi k t}{a}\right) + b_k \sin\left(\frac{2\pi k t}{a}\right) \right],$$

then
$$a_k = \frac{2}{a} \int_0^a p(t) \cos\left(2\pi k \frac{t}{a}\right) dt$$

$$b_k = \frac{2}{a} \int_0^a p(t) \sin\left(2\pi k \frac{t}{a}\right) dt$$

We have considered only functions $p(t)$ from T_N. Now we have a question:

Is it possible to write

$$f(t) = \sum_{k=-N}^{N} c_k e^{2\pi i k \frac{t}{a}} \quad \text{for}$$

any periodic function $f(t)$ with period $a>0$ (i.e. also for $f(t) \notin T_N$) ?

The answer is:

<u>NO</u> for N finite

<u>YES</u> for $N \to \infty$, then instead of space T_N we use the space $L^2_p(0,a)$, which contains periodic functions with period $a>0$ and for which the integral $\int_0^a |f(t)|^2 dt$ exists and is finite. The scalar product and norm are in space $L^2_p(0,a)$ the same like in T_N.

However in real applications we cannot use $N \to \infty$. Let's try to find the best approximation of $f(t) \in L_p^2(0,a)$ by the polynomial $p(t) \in T_N$ in sense of least square method, i.e. we search coefficients x_K for minimum of norm $\| f(t) - p(t) \|_2$ where $p(t) = \sum_{K=-N}^{N} x_K e_K(t)$

$$\| f - p \|_2^2 = (f-p, f-p) = (f, f-p) - (p, f-p) =$$

$$= (f,f) - (f,p) - (p,f) + (p,p) =$$

$$= \| f \|_2^2 - (f,p) - (p,f) + \| p \|_2^2 ,$$

where

$$\| p \|_2^2 = \left(\sum_{K=-N}^{N} x_K e_K(t) , \sum_{K=-N}^{N} x_K e_K(t) \right) =$$

$$= a \sum_{K=-N}^{N} | x_K |^2$$

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$$(f(t), p(t)) = \left(f(t), \sum_{k=-N}^{N} x_k e_k(t) \right) =$$

$$= \sum_{k=-N}^{N} \overline{x_k} \, (f(t), e_k(t)) = \sum_{k=-N}^{N} \overline{x_k} \, a \, c_k$$

where $\quad c_k = \frac{1}{a} (f(t), e_k(t))$

$$(p(t), f(t)) = \left(\sum_{k=-N}^{N} x_k e_k(t), f(t) \right) =$$

$$= \sum_{k=-N}^{N} x_k \, (e_k(t), f(t)) = \sum_{k=-N}^{N} x_k \, a \, \overline{c_k}$$

let's go back to norm

$$\underline{\underline{\|f-p\|_2^2}} = \|f\|_2^2 - \sum_{k=-N}^{N} \overline{x_k} \, a \, c_k - \sum_{k=-N}^{N} x_k \, a \, \overline{c_k} +$$

$$+ a \sum_{k=-N}^{N} |x_k|^2 = \underline{\underline{\|f\|_2^2 +}}$$

$$\underline{\underline{+ a \sum_{k=-N}^{N} \left(-\overline{x_k} c_k - x_k \overline{c_k} + |x_k|^2 \right)}}$$

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It is possible to show that

$$\bar{x}c + x\bar{c} = (\mathrm{Re}\,x - i\,\mathrm{Im}\,x)(\mathrm{Re}\,c + i\,\mathrm{Im}\,c) +$$
$$+ (\mathrm{Re}\,x + i\,\mathrm{Im}\,x)(\mathrm{Re}\,c - i\,\mathrm{Im}\,c) = \dots =$$
$$= 2(\mathrm{Re}\,x\,\mathrm{Re}\,c + \mathrm{Im}\,x\,\mathrm{Im}\,c)$$

and that

$$|c-x|^2 = (\mathrm{Re}\,c - \mathrm{Re}\,x)^2 + (\mathrm{Im}\,c - \mathrm{Im}\,x)^2 =$$
$$= \dots = (\mathrm{Re}\,c)^2 + (\mathrm{Im}\,c)^2 + (\mathrm{Re}\,x)^2 + (\mathrm{Im}\,x)^2 -$$
$$-2(\mathrm{Re}\,c\,\mathrm{Re}\,x + \mathrm{Im}\,c\,\mathrm{Im}\,x) = |c|^2 + |x|^2 -$$
$$- \bar{x}c - x\bar{c}$$

$$\Rightarrow \quad -\bar{x}_k c_k - x_k \bar{c}_k + |x_k|^2 = |c_k - x_k|^2 - |c_k|^2$$

$$\Rightarrow \quad \|f-p\|_2^2 = \|f\|_2^2 + a \sum_{k=-N}^{N} \left(|c_k - x_k|^2 - |c_k|^2\right) (*)$$

is minimal for $\underline{x_k = c_k}$

It means that the best approximation of $f(t)$ in T_N is $f_N(t)$

$$f_N(t) = \sum_{k=-N}^{N} c_k e^{2\pi i k \frac{t}{a}}$$

with
$$c_k = \frac{1}{a}\int_0^a f(t) e^{-2\pi i k \frac{t}{a}} dt .$$

Bessel inequality

Let's consider the equation (*) with $x_k = c_k$ and $p = f_N$, i.e.

$$\|f - f_N\|_2^2 = \|f\|_2^2 - a\sum_{k=-N}^{N} |c_k|^2$$

$$a\sum_{k=-N}^{N} |c_k|^2 + \|f-f_N\|_2^2 = \|f\|_2^2$$

$$\Rightarrow \boxed{\sum_{k=-N}^{N} |c_k|^2 \leq \frac{1}{a}\int_0^a |f(t)|^2 dt}$$

Corolary: $\int_0^a |f(t)|^2 dt$ is bounded, because $f(t) \in L_p^2(0,a) \Rightarrow$ also for $N \to \infty$ $\sum_{k=-\infty}^{\infty} |c_k|^2$ must be bounded $\Rightarrow$

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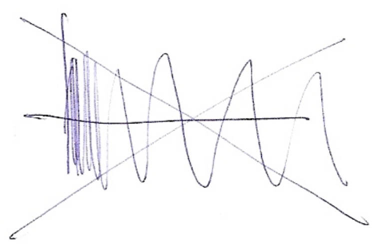
$$\Rightarrow \quad \lim_{k \to \infty} |c_k| = 0$$

Remark: For $N \to \infty$ it is possible to show that
$$\int_0^a |f(t) - f_N(t)|^2 \, dt \to 0 ,$$

i.e. that $f_N(t)$ converges to $f(t)$ in average.

Theorem: If $f(t) \in L_p^2(0,a)$ satisfies the Dirichlet conditions, i.e.

1) it is periodic with period $a > 0$
2) it is piecewise continuous (with finite number of discontinuities)
3) it is piecewise monotonic (it is growing or decreasing on finite number of intervals) "not like this"



then

$$S(t_0) = \frac{1}{2}\left(\lim_{t \to t_0-} f(t) + \lim_{t \to t_0+} f(t)\right),$$

where $s(t)$ is the sum of Fourier series

$$s(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos\left(\frac{2\pi k t}{a}\right) + b_k \sin\left(\frac{2\pi k t}{a}\right)\right)$$

with
$$a_k = \frac{2}{a}\int_0^a f(t)\cos\left(\frac{2\pi k t}{a}\right)dt$$

$$b_k = \frac{2}{a}\int_0^a f(t)\sin\left(\frac{2\pi k t}{a}\right)dt$$

$\square$

Parseval equality

It is possible to show that

$$\lim_{N\to\infty}(f_N, f_N) = \boxed{\sum_{k=-\infty}^{\infty}|c_k|^2 = \frac{1}{a}\int_0^a |f(t)|^2 dt}$$

This is very important property. If $f(t)$ is some signal, then $\frac{1}{a}\int_0^a |f(t)|^2 dt$ corresponds to the energy of signal.

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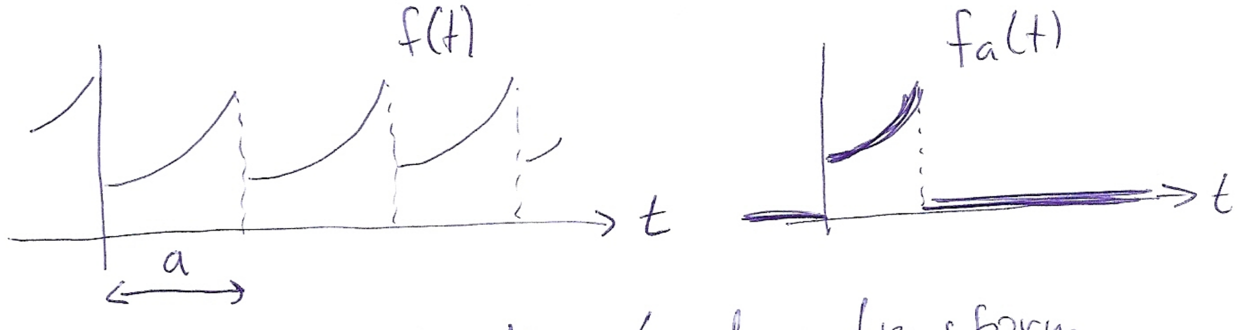
The Fourier series for function $f(t) \in L_p^2$ $(0,a)$ can be understood as a sum of harmonic signals with frequency $\frac{k}{a}$ and amplitude c_k. Parseval equality states, that the decomposition of $f(t)$ into Fourier series keeps the energy of signal (of course for $N \to \infty$).

Remark: Let's consider the periodic function $f(t)$ with period $a > 0$ and denote one period of $f(t)$ by $f_a(t)$, i.e.



we know that the Laplace transform of $f_a(t)$ is $F_a(s) = \mathcal{L}\{f_a(t)\} = \int_0^a f_a(t)\, e^{-st}\, dt$

and the Fourier coefficient c_k is defined $c_k = \frac{1}{a} \int_0^a f(t)\, e^{-2\pi i k \frac{t}{a}}\, dt$

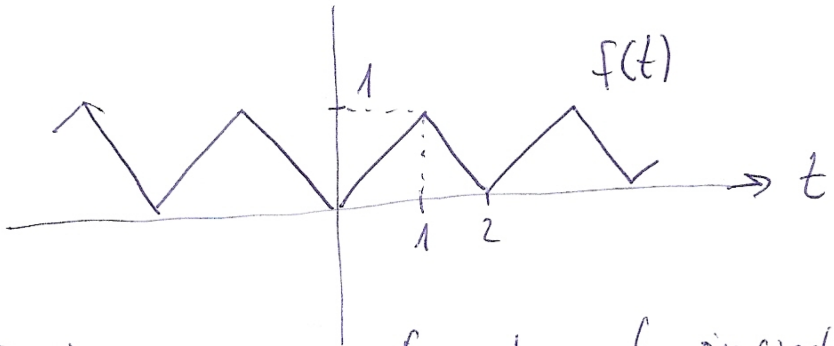
(120)

$$\Rightarrow \quad \underline{c_k = \frac{1}{a} F_a\left(2\pi i \frac{k}{a}\right)} \quad ,$$

i.e. the Fourier coefficients can be computed using dictionary of Laplace transform for "one period" with substitution

$$s = \frac{2\pi i k}{a} .$$

Exercise: Find the Fourier series for periodic function $f(t)$ with period $a=2$



$f(t)$ is even function (in czech sudá funkce)

$$\Rightarrow \quad b_k = 0$$

$$a_k = \frac{2}{2} \int_{-1}^{1} f(t) \cos\left(\frac{2\pi k t}{2}\right) dt =$$

$$= 2 \int_{0}^{1} f(t) \cos(\pi k t)\, dt \quad , \quad k = 1, 2, \dots$$

$$a_0 = 2 \int_{0}^{1} f(t)\, dt = 1$$

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```
> a0:=2*int(t,t=0..1);
```

$$a0 := 1$$

```
> ak:=2*int(t*cos(Pi*k*t),t=0..1);
```

$$ak := \frac{2\,(-1+\cos(\pi\,k)+\sin(\pi\,k)\,\pi\,k)}{\pi^2\,k^2}$$

```
> s2:=a0/2+sum(ak*cos(Pi*k*t),k=1..2);
```

$$s2 := \frac{1}{2} - \frac{4\cos(\pi\,t)}{\pi^2}$$

```
> s4:=a0/2+sum(ak*cos(Pi*k*t),k=1..4);
```

$$s4 := \frac{1}{2} - \frac{4\cos(\pi\,t)}{\pi^2} - \frac{4\cos(3\,\pi\,t)}{9\,\pi^2}$$

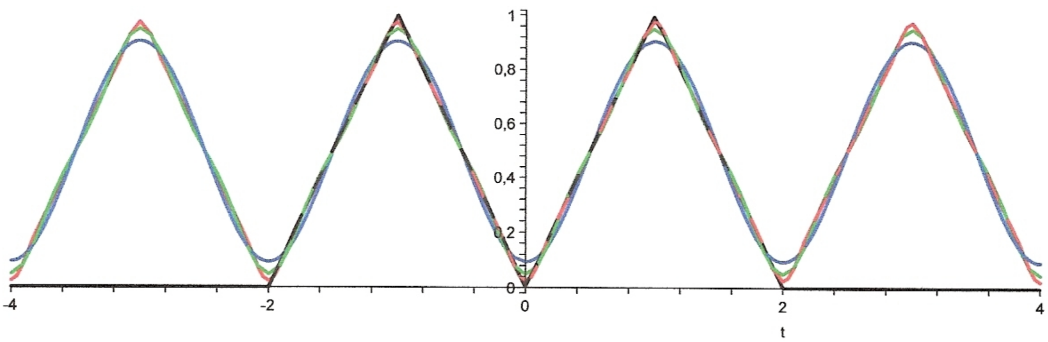
```
> s8:=a0/2+sum(ak*cos(Pi*k*t),k=1..8);
```

$$s8 := \frac{1}{2} - \frac{4\cos(\pi\,t)}{\pi^2} - \frac{4\cos(3\,\pi\,t)}{9\,\pi^2} - \frac{4\cos(5\,\pi\,t)}{25\,\pi^2} - \frac{4\cos(7\,\pi\,t)}{49\,\pi^2}$$

```
> f:=piecewise(t>-2 and t<-1, t+2, t>-1 and t< 0, -t, t>0 and t<1, t, t>
  and t<2, -t+2);
```

$$f := \begin{cases} t+2 & -2 < t \textbf{ and } t < -1 \\ -t & -1 < t \textbf{ and } t < 0 \\ t & 0 < t \textbf{ and } t < 1 \\ -t+2 & 1 < t \textbf{ and } t < 2 \end{cases}$$

```
> with(plots):  p1:=plot(f,t=-4..4,color=black):
> p2:=plot(s2,t=-4..4,color=blue):  p3:=plot(s4,t=-4..4,color=green):
  p4:=plot(s8,t=-4..4,color=red):
> display(p1,p2,p3,p4);
```



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Let's "check" the Parseval equality

$$\sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{1}{a}\int_0^a |f(t)|^2\, dt$$

$$\text{difference} = \frac{1}{a}\int_0^a |f(t)|^2\, dt - \sum_{k=-N}^{N} |c_k|^2 =$$

$$= \frac{2}{2}\int_0^1 t^2\, dt - \sum_{k=-N}^{N} |c_k|^2 \,,$$ where

$$b_k = i\,(c_k - c_{-k}) = 0 \Rightarrow c_k = c_{-k}$$

$$a_k = c_k + c_{-k} = 2\,c_k \Rightarrow \boxed{c_k = \frac{a_k}{2}}$$

see the "difference" for N = 2, 4, 8, 16, 32

```
> difference2:=evalf(int(t*t,t=0..1)-(a0/2)^2-2*sum((ak/2)^2,k=1..2));
```

difference2 := 0.00120547533

```
> difference4:=evalf(int(t*t,t=0..1)-(a0/2)^2-2*sum((ak/2)^2,k=1..4));
```

difference4 := 0.00019155116

```
> difference8:=evalf(int(t*t,t=0..1)-(a0/2)^2-2*sum((ak/2)^2,k=1..8));
```

difference8 := 0.00002594090

```
> difference16:=evalf(int(t*t,t=0..1)-(a0/2)^2-2*sum((ak/2)^2,k=1..16));
```

difference16 := 0.00000331607

```
> difference32:=evalf(int(t*t,t=0..1)-(a0/2)^2-2*sum((ak/2)^2,k=1..32));
```

difference32 := $4.1694\ 10^{-7}$

<u>Remark:</u> Physical meaning of Fourier coefficients:

note that $a_k \cos\left(\frac{2\pi k t}{a}\right) + b_k \sin\left(\frac{2\pi k t}{a}\right) =$

$$= A_k \sin\varphi_k \cos\left(\frac{2\pi k t}{a}\right) + A_k \cos\varphi_k \sin\left(\frac{2\pi k t}{a}\right) =$$

$$= A_k \sin\left(\frac{2\pi k t}{a} + \varphi_k\right) = A_k \sin(\omega_k t + \varphi_k)$$

where

$$A_k \sin\varphi_k = a_k$$
$$A_k \cos\varphi_k = b_k$$

$$\Rightarrow A_k = \sqrt{a_k^2 + b_k^2}$$

$$\operatorname{tg}\varphi_k = \frac{a_k}{b_k}, \text{ i.e. } \varphi_k = \operatorname{arctg}\frac{a_k}{b_k}$$

i.e. $$f(t) \doteq \frac{a_0}{2} + \sum_{k=1}^{N} A_k \sin(\omega_k t + \varphi_k)$$

i.e. the Fourier series decomposes "a signal" $f(t)$ into sum of harmonic signals with frequencies $\nu_k = \frac{k}{a}$, $k = 1, \dots, N$ (theoretically $N \to \infty$) or angular velocities ω_k. φ_k is the phase shift for k-th frequency and A_k is the amplitude. We see that Fourier series has discrete spectra of frequencies.