Solution of initial and mixed problems for partial differential equations

Example: Consider a mixed problem (initial-boundary value problem) for wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad a > 0,$$

$$[x,t] \in (0, +\infty) \times (0, +\infty)$$

with initial conditions

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0, \quad x \in \langle 0; +\infty)$$

and boundary conditions

$$u(0,t) = \begin{cases} \sin(2\pi t), & t \in \langle 0,1 \rangle \\ 0, & t > 1 \end{cases}$$

and we expect bounded solution.

Now we would like to use Laplace transform, to do it we have to choose proper variable $\rightarrow t$, because we have both initial conditions $u(x,0) = 0$ and $\frac{\partial u}{\partial t}(x,0) = 0$

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$$\mathcal{L}\left\{\frac{\partial^2 u(x,t)}{\partial t^2}\right\} = s^2\mathcal{L}\{u(x,t)\} - s\,u(x,0) -$$

$$- \frac{\partial u}{\partial t}(x,0) = s^2 U(x,s)$$

where $U(x,s) = \mathcal{L}\{u(x,t)\} = \int_0^\infty u(x,t)e^{-st}dt$

$$\mathcal{L}\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} = \frac{\partial^2 U(x,s)}{\partial x^2}$$

$\Rightarrow$ we get equation

$$s^2 U(x,s) = a^2 \frac{\partial^2 U(x,s)}{\partial^2 x},$$

which includes initial conditions. This equation can be understood as ordinary differential equation with some parameter s, i.e.

$$a^2 U'' - s^2 U = 0$$

char. eq.: $a^2\lambda^2 - s^2 = 0$

$$\lambda_{1,2} = \pm s/a$$

$$\Rightarrow \underline{U(x,s)} = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} =$$

$$= \underline{c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x}}$$

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$U(x,s)$ has to satisfy two conditions

1) $U(x=0, s) = \mathcal{L}\{u(0,t)\}$

where $u(0,t) = \sin(2\pi t)\cdot\left(h(t) - h(t-1)\right)$

($h(t)$ ... is Heaviside function)

2) the solution should be bounded for $[x,t] \in (0,\infty)\times(0,\infty)$.

As $U(x,s)$ exists for $\operatorname{Re} s > 0$ and $a>0$ therefore $C_1 = 0$

(note $e^{\frac{s}{a}x} \to \infty$ for $x \to \infty$)

$$U(0,s) = c_2 e^{-\frac{s}{a}\cdot 0} = c_2 = \mathcal{L}\left\{\sin(2\pi t)\left(h(t)-h(t-1)\right)\right\}$$

see Maple file

solution is $\underline{\underline{u(x,t) = \mathcal{L}^{-1}\left\{c_2 e^{-\frac{s}{a}x}\right\}}}$

see graph in Maple file

(74) $a=1$ is considered

```
> restart:with(inttrans):
> c2:=laplace(sin(2*Pi*t)*(Heaviside(t)-Heaviside(t-1)),t,s);
```

$$c2 := \frac{2\,\pi\,(1-e^{(-s)})}{s^2+4\,\pi^2}$$

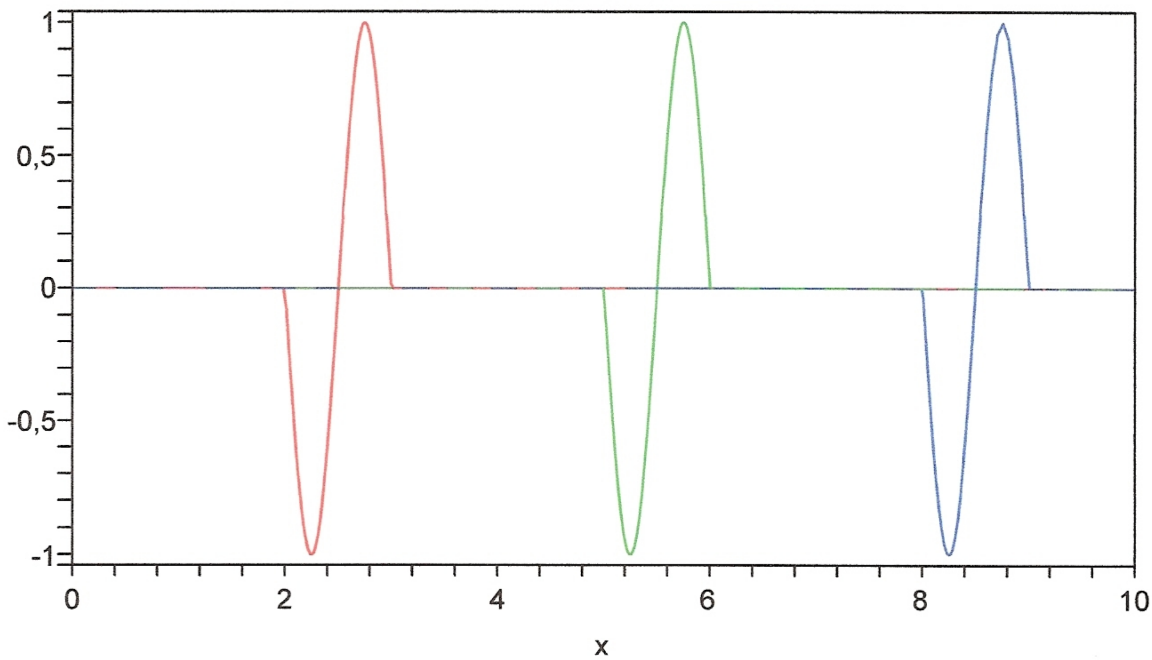
```
> U(x,s):=c2*exp(-s*x);
```

$$U(x,s) := \frac{2\,\pi\,(1-e^{(-s)})\,e^{(-s\,x)}}{s^2+4\,\pi^2}$$

```
> u(x,t):=invlaplace(U(x,s),s,t);
```

$$u(x,t) := 2\,\pi\left(\mathit{invlaplace}\left(\frac{e^{(-s\,x)}}{s^2+4\,\pi^2}, s, t\right) - \mathit{invlaplace}\left(\frac{e^{(-(x+1)\,s)}}{s^2+4\,\pi^2}, s, t\right)\right)$$

```
> with(plots):
Warning, the name changecoords has been redefined
> p1:=plot(subs(t=3,u(x,t)),x=0..10,color=red):
> p2:=plot(subs(t=6,u(x,t)),x=0..10,color=green):
> p3:=plot(subs(t=9,u(x,t)),x=0..10,color=blue):
> display({p1,p2,p3},axes=boxed);
```



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Another example:

Consider a mixed problem for heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2},$$

$$[x,t] \in (0,1) \times (0,\infty)$$

with initial condition

$$u(x,0) = 1 + \sin(\pi x), \quad x \in \langle 0; 1 \rangle$$

and boundary conditions

$$u(0,t) = u(1,t) = 1, \quad t \geq 0$$

→ we perform Laplace transform again in variable t (initial conditions are given only for t)

$$s\,U(x,s) - u(x,0) = \frac{\partial^2 U(x,s)}{\partial x^2},$$

where $$U(x,s) = \mathcal{L}\{u(x,t)\} = \int_0^\infty u(x,t)\, e^{-st}\, dt$$

so $$\frac{\partial^2 U(x,s)}{\partial x^2} - s\,U(x,s) = -1 - \sin(\pi x),$$

which represent ordinary differential equation in variable x. Solution $U(x,s)$ is obtained by Maple. Solution $U(x,s)$ has to satisfy boundary conditions, i.e.

$$U(0,s) = \mathcal{L}\{u(0,t)\} = \mathcal{L}\{1\} = \frac{1}{s}$$

$$U(1,s) = \mathcal{L}\{u(1,t)\} = \mathcal{L}\{1\} = \frac{1}{s}$$

→ terms _F1(s) and _F2(s) in Maple file are equal to zero

from Maple $U(x,s) = \dfrac{s+\pi^2+s\cdot\sin(\pi x)}{s\cdot(s+\pi^2)}$

$$u(x,t) = \mathcal{L}^{-1}\{U(x,s)\}$$

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```
> restart:with(inttrans):with(plots):
Warning, the name changecoords has been redefined
```

```
> ode:=diff(U(x,s),x,x)-s*U(x,s)=-1-sin(Pi*x);
```

$$ode := \left(\frac{\partial^2}{\partial x^2} U(x, s)\right) - s\,U(x, s) = -1 - \sin(\pi x)$$

```
> res(x,s):=dsolve(ode,U(x,s));
```

$$res(x, s) := U(x, s) = e^{(\sqrt{s}\,x)}\,\_F2(s) + e^{(-\sqrt{s}\,x)}\,\_F1(s) + \frac{s + \pi^2 + s\sin(\pi x)}{s\,(s + \pi^2)}$$

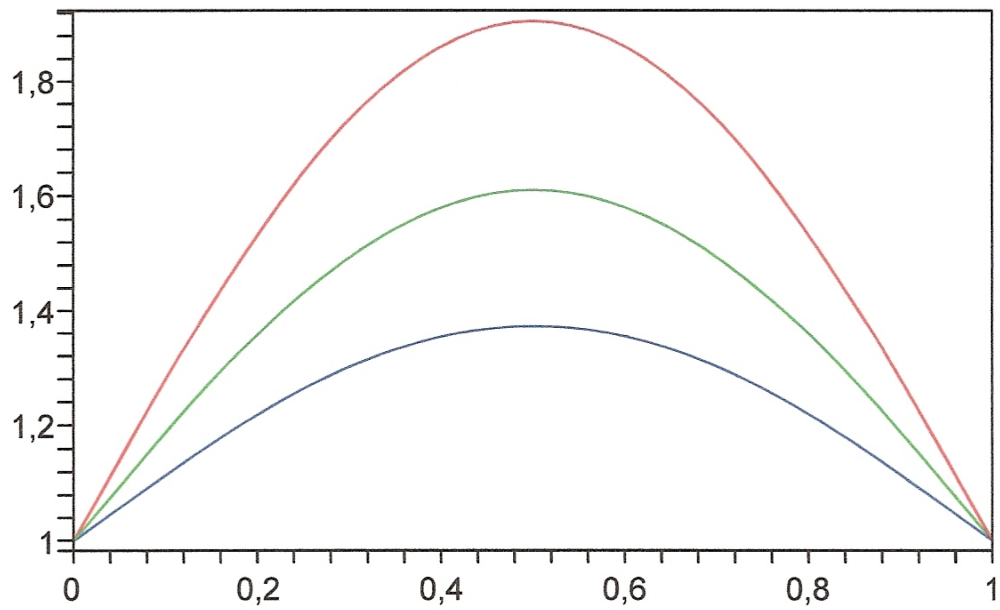
```
> res1(x,s):=rhs(subs(_F2(s)=0,_F1(s)=0,res(x,s)));
```

$$res1(x, s) := \frac{s + \pi^2 + s\sin(\pi x)}{s\,(s + \pi^2)}$$

```
> res2(x,t):=invlaplace(res1(x,s),s,t);
```

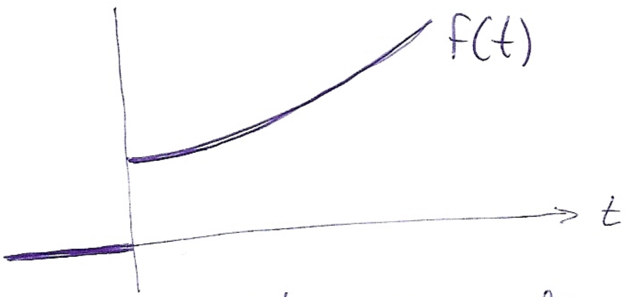
$$res2(x, t) := \sin(\pi x)\,e^{(-\pi^2 t)} + 1$$

```
> p1:=plot(subs(t=0.01,res2(x,t)),x=0..1,color=red):
> p2:=plot(subs(t=0.05,res2(x,t)),x=0..1,color=green):
> p3:=plot(subs(t=0.1,res2(x,t)),x=0..1,color=blue):
> display({p1,p2,p3},axes=boxed);
```

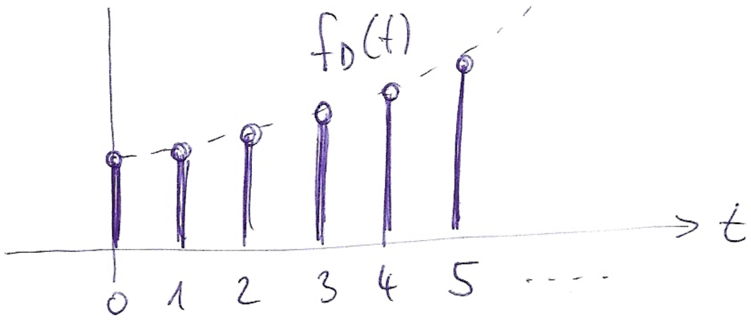


DISCRETE LAPLACE TRANSFORM, Z - TRANSFORM

Consider continuous function $f(t) \in T_\alpha$



Let's further consider sampling with frequency equal to one (the following idea can be written easily for any frequency). The discrete "version" of f(t) is denoted $f_D(t)$



The Laplace transform of $f_D(t)$ can be computed as

$$\underline{\underline{\mathcal{L}\{f_D(t)\}}} = \int_0^\infty f_D(t) e^{-st} \, dt =$$

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$$= \int_0^\infty \left(\sum_{k=0}^{\infty} f(k)\,\delta(t-k) \right) e^{-st}\,dt =$$

$$= \sum_{k=0}^{\infty} f(k) \int_0^\infty \delta(t-k)\,e^{-st}\,dt = \sum_{k=0}^{\infty} f(k)\,\mathcal{L}\{\delta(t-k)\} =$$

$$= \sum_{k=0}^{\infty} f(k)\,e^{-ks}\,\mathcal{L}\{\delta(t)\} = \underline{\underline{\sum_{k=0}^{\infty} f(k)\,e^{-ks}}}$$

$$\left(\text{where } \mathcal{L}\{\delta(t)\} = \int_0^\infty \delta(t)\,e^{-st}\,dt = e^{-s\cdot 0} = 1 \right)$$

We see that the <u>discrete Laplace transform of sequence</u> (in czech "posloupnost") <u>f(k)</u> is $\underline{\sum_{k=0}^{\infty} f(k)\,e^{-ks}}$.

<u>Remark:</u> Note that

$$\int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} f(k)\,\delta(t-k) \right) e^{-i\omega t}\,dt = \cdots\cdots =$$

$$= \sum_{k=-\infty}^{\infty} f(k)\,e^{-i\omega k}$$ represents the discrete Fourier transform of sequence $f(k)$

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(we already know that variable s in Laplace transform corresponsds to variable $i\omega$ in Fourier transform) $\square$

If we substitute $z = e^{s}$ in discrete Laplace transform we get following correspondence

$$\underbrace{\{f(k)\} \triangleq \sum_{k=0}^{\infty} f(k)\, z^{-k}}_{}$$

which is the definition of Z-transform

<u>Definition:</u> If Laurent series $\sum_{k=0}^{\infty} \frac{f_K}{z^k}$ converges, then $F(z) = \sum_{K=0}^{\infty} \frac{f_K}{z^k}$ is the Z-transform of sequence $\{f_k\}$, where $f_k = f(k)$.

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<u>Exercise:</u> Find z-transform of sequence

$$\{f_k\} = \{2^k\}_{k=0}^{\infty}.$$

$$F(z) = \sum_{k=0}^{\infty} \frac{2^K}{z^K} = \sum_{k=0}^{\infty} \left(\frac{2}{z}\right)^k =$$

$$= \frac{1}{1-\frac{2}{z}} = \frac{1}{\frac{z-2}{z}} = \underline{\underline{\frac{z}{z-2}}}$$

for $\left|\frac{2}{z}\right| < 1$ i.e. for $\underline{\underline{|z|>2}}$

in Maple:

```
with(inttrans):
simplify(ztrans(2^K,K,z));
invztrans(z/(z-2),z,K);
```

<u>Remark:</u> the discrete Laplace transform of sequence $\{f_k\}=\{2^k\}_{k=0}^{\infty}$ would be

$$\mathcal{L}_D\{f_k\} = \frac{e^s}{e^s-2}.$$

PROPERTIES OF Z-TRANSFORM

1) Existence of Z-transform

If there exist such $M \geq 0$ and $\alpha \in \mathbb{R}$ that for every a_k, $k \geq k_0$ of sequence $\{a_n\}_{n=0}^{\infty}$ holds $|a_n| \leq M e^{\alpha k}$, then the serie $\sum_{k=0}^{\infty} \frac{a_k}{z^k}$ is convergent for $|z| > e^{\alpha}$.

Proof:
$$\sum_{k=0}^{\infty} \frac{a_k}{z^k} \leq \sum_{k=0}^{\infty} \left| \frac{a_k}{z^k} \right| \leq \sum_{k=0}^{\infty} \frac{M e^{\alpha k}}{|z|^k} =$$

$$= M \underbrace{\sum_{k=0}^{\infty} \left(\frac{e^{\alpha}}{|z|} \right)^k}_{\text{this series is convergent for } |z| > e^{\alpha}}$$

$\Rightarrow$ series $\sum_{k=0}^{\infty} \frac{a_k}{z^k}$ is also convergent for $|z| > e^{\alpha}$

$\square$

2) Linearity of Z-transform

$$\mathcal{Z}\{c_1 a_k + c_2 b_k\} = c_1 \mathcal{Z}\{a_k\} + c_2 \mathcal{Z}\{b_k\}$$

3) Z-transform of shifted sequence

$$\mathcal{Z}\{f_{k+m}\} = \sum_{k=0}^{\infty} \frac{f_{k+m}}{z^k} = z^m \sum_{k=m}^{\infty} \frac{f_k}{z^k} =$$

$$= z^m \left(\sum_{k=0}^{\infty} \left(\frac{f_k}{z^k}\right) - f_0 - \frac{f_1}{z} - \frac{f_2}{z^2} - \dots - \frac{f_{m-1}}{z^{m-1}} \right) =$$

$$= z^m \mathcal{Z}\{f_k\} - z^m f_0 - z^{m-1} f_1 - \dots - z f_{m-1}$$

4) Inverse transform

we know that Z-transform is defined as

$$\mathcal{Z}\{f_k\} = F(z) = \sum_{k=0}^{\infty} \frac{f_k}{z^k} \quad , \quad \text{where}$$

$F(z)$ is known and we search f_k, i.e. we try to write Laurent series for $F(z)$

coefficients f_k can be computed e.g. using residues or geometric series formula.

Example: $F(z) = \frac{z}{z-a}$ is given.

Find $\{f_k\} = \mathcal{Z}^{-1}\{F(z)\}$

a) using "geometric series formula"

$$\frac{z}{z-a} = \frac{1}{1-\frac{a}{z}} = \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k \rightarrow \underline{\underline{f_k = a^k}}$$

$$\left|\frac{a}{z}\right| < 1 \rightarrow |z| > |a|$$

b) using residues: we know that residue is the "f_1" coefficient.

Function $F(z) = \frac{z}{z-a}$ has singular point in $z=a$ $\Rightarrow$

$$f_1 = \text{Res}\left[\frac{z}{z-a}, z=a\right]$$

$$f_2 = \text{Res}\left[\frac{z}{z-a}\cdot z, z=a\right]$$

$$\vdots$$

$$f_k = \text{Res}\left[\frac{z^k}{z-a}, z=a\right]$$

by Maple

```
readlib(residue):
residue(z^K/(z-a), z=a);
```

$\longrightarrow a^K$

5) Convolution

The convolution of two sequences $\{f_n\}$ and $\{g_n\}$ is defined as

$$\{f_n\} * \{g_n\} = \{f_n * g_n\} =$$

$$= \left\{ \sum_{K=0}^{n} f_K \, g_{n-K} \right\}_{n=0}^{\infty}$$

also for sequences comutative law holds, i.e.

$$\{f_n\} * \{g_n\} = \{g_n\} * \{f_n\}$$

Z-transform of convolution is

$$\mathcal{Z}\{f_n * g_n\} = \mathcal{Z}\{f_n\} \cdot \mathcal{Z}\{g_n\}.$$

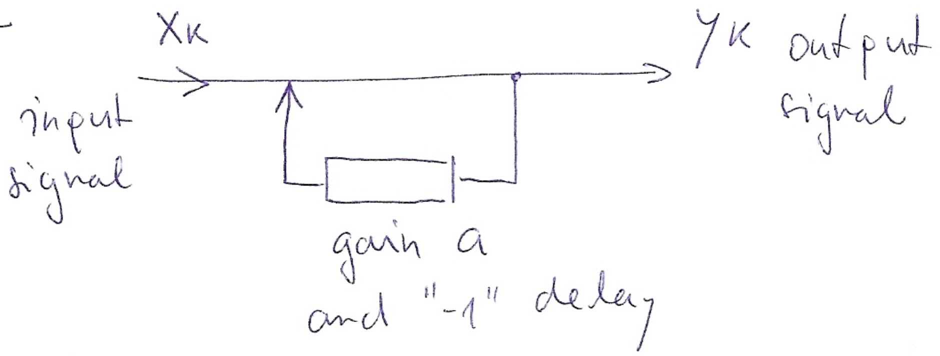
Proof: $\mathcal{Z}\{f_n\} \cdot \mathcal{Z}\{g_n\} =$

$$= \left(f_0 + \frac{f_1}{z} + \frac{f_2}{z^2} + \cdots \right) \cdot \left(g_0 + \frac{g_1}{z} + \frac{g_2}{z^2} + \cdots \right) =$$

$$= f_0 g_0 + \frac{f_0 g_1 + f_1 g_0}{z} + \frac{f_0 g_2 + f_1 g_1 + f_2 g_0}{z^2} +$$

$$+ \cdots = \mathcal{Z}\{ f_n * g_n \} \qquad \square$$

Examples: discrete filter



The filter is given as $y_{k+1} = a y_k + x_{k+1}$

Let's do Z-transform of this equation

$$z Y(z) - z y_0 = a Y(z) + z X(z) - z x_0$$

consider $x_0 = y_0 = 0$

(87)

$$(z-a)\,Y(z) = z\,X(z)$$

$$Y(z) = \underbrace{\frac{z}{z-a}}_{\text{transfer function } H(z)} X(z)$$

from previous example we know that

$$\mathcal{Z}^{-1}\{H(z)\} = \{a^k\}_{k=0}^{\infty}$$

therefore $\{y_k\} = \{a^k\} * \{x_k\}$ for any input sequence $\{x_k\}$.

Remarks: If we use $\{x_k\} = \{1, 0, 0, 0, \dots\}$ in previous example, where sequence $\{1, 0, 0, 0, \dots\}$ is called Cronecker's delta

$$\delta_0 = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}$$

(it is discrete analogy of Dirac δ function), then

$\mathcal{Z}\{\delta_0\} = 1$ and the output directly gives the transfer function since

$$Y(z) = H(z) \cdot \underbrace{1}_{= \mathcal{Z}\{\delta_0\}}$$

i.e. $\underline{\{h_k\}} = \mathcal{Z}^{-1}\{H(z)\} = \underline{\{y_k\}}$

6) z-transform of diferences

consider a differentiable function $f(t)$ and sampling with frequency $= 1$ (for simplicity), i.e. we get the discrete function (or sequence) $f_k = f(k)$.
We define "the 1-st difference" as

$$\underline{\Delta^1 f_k = f_{k+1} - f_k} = \frac{f(k+1) - f(k)}{k+1 - k} \approx \frac{df}{dt}$$

the "2-nd difference" as

$$\underline{\Delta^2 f_k} = \Delta^1(\Delta^1 f_k) = \Delta^1(f_{k+1} - f_k) =$$

$$= \Delta^1 f_{k+1} - \Delta^1 f_k = \underline{f_{k+2} - 2 f_{k+1} + f_k} \approx$$

$$\approx \frac{d}{dt}\left(\frac{df}{dt}\right) = \frac{d^2 f}{dt^2}$$