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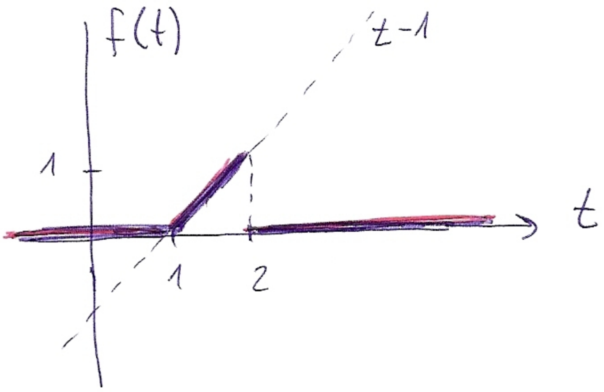
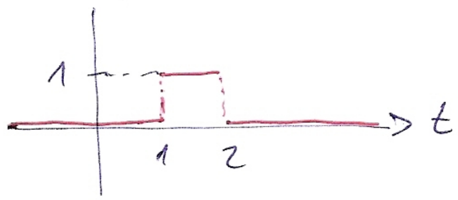
Exercise: (initial value problem for ODE with discontinuous right had side)

Consider
$$\ddot{x}(t) + x(t) = f(t),$$
$$x(0) = 0, \quad \dot{x}(0) = 0 \quad \text{and}$$



$$f(t) = (t-1) \cdot \underbrace{\left(h(t-1) - h(t-2) \right)}$$



$$\mathcal{L} \,\Big|\; \ddot{x}(t) + x(t) = f(t)$$

$$s^2 X(s) + X(s) = \mathcal{L}\{f(t)\}$$

$$X(s) = \frac{\mathcal{L}\{f(t)\}}{s^2+1}$$

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note here is "p" instead of "s"

```
> with(inttrans):
> F(p):=laplace((t-1)*(Heaviside(t-1)-Heaviside(t-2)),t,p);
```

$$F(p) := \frac{e^{(-p)} - e^{(-2\,p)}\,(p+1)}{p^2}$$

```
> X(p):=F(p)/(p*p+1);
```

$$X(p) := \frac{e^{(-p)} - e^{(-2\,p)}\,(p+1)}{p^2\,(p^2+1)}$$

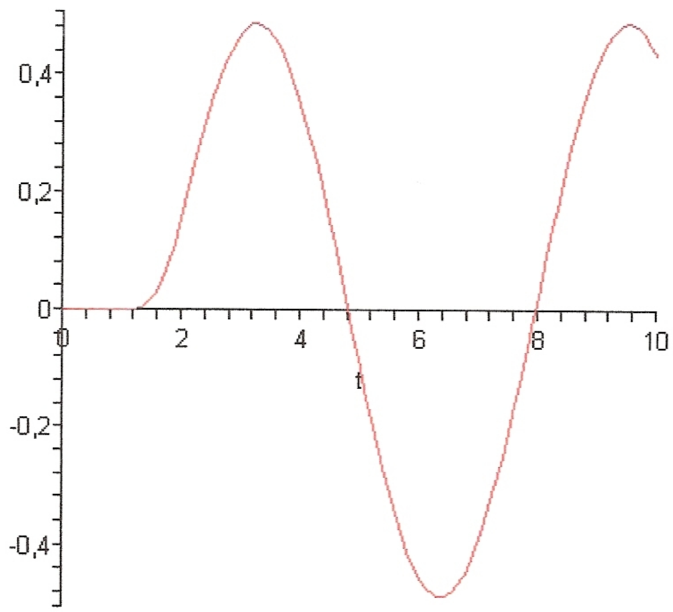
```
> x(t):=invlaplace(X(p),p,t);
```

$$x(t) := \mathrm{Heaviside}(t-1)\,(t-1-\sin(t-1)) - \mathrm{Heaviside}(t-2)\left(-2+2\sin\left(\frac{1}{2}t-1\right)^2 - \sin(t-2)+t\right)$$

```
> with(plots):
```

Warning, the name changecoords has been redefined

```
> plot(x(t),t=0..10);
```




```
>
```

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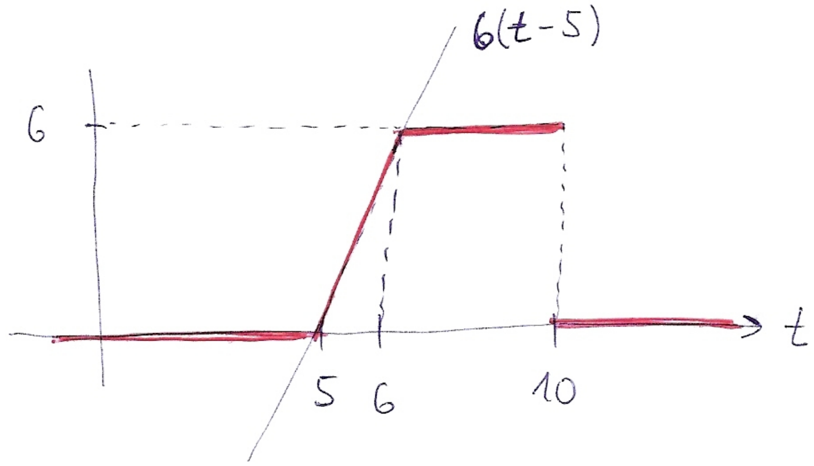
Exercise: Describe the function

$$f(t) = \begin{cases} 0, & t < 5 \\ 6(t-5), & t \in (5,6) \\ 6, & t \in (6,10) \\ 0, & t > 10 \end{cases}$$

by means of Heaviside function.



$$f(t) = 6(t-5) \cdot \left[h(t-5) - h(t-6) \right] +$$

$$+ 6 \cdot \left[h(t-6) - h(t-10) \right]$$

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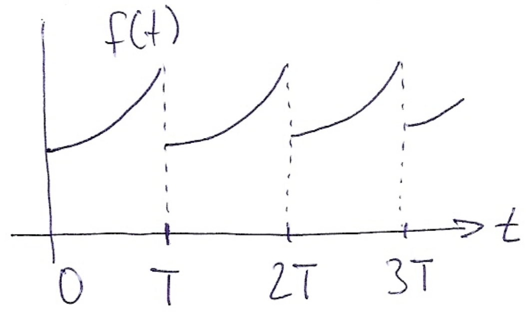
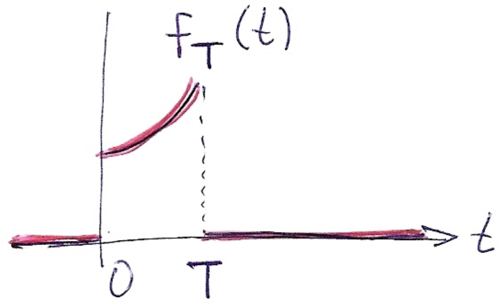
7) Laplace transform of periodic function



$f(t)$ is periodic with period $T>0$



$f_T(t)$ is "one period"

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} f(t)\, e^{-st}\, dt =$$

$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} f(t-nT)\, e^{-st}\, dt = \left[\begin{array}{l} \text{substitute} \\ \tau = t - nT \\ d\tau = dt \end{array}\right] =$$

$$= \sum_{n=0}^{\infty} \int_0^T f(\tau)\, e^{-s(\tau + nT)}\, d\tau =$$

$$= \int_0^T f(\tau)\, e^{-s\tau}\, d\tau \cdot \sum_{n=0}^{\infty} e^{-snT} =$$

$$= \mathcal{L}\{f_T\} \cdot \sum_{n=0}^{\infty} \left(e^{-sT}\right)^n =$$

$$= \mathcal{L}\{f_T\} \cdot \frac{1}{1-e^{-sT}}, \quad \operatorname{Re} s > 0$$

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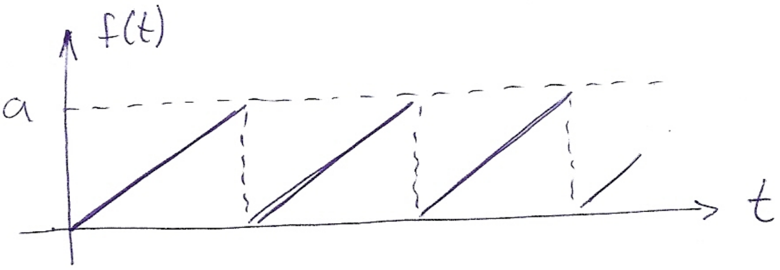
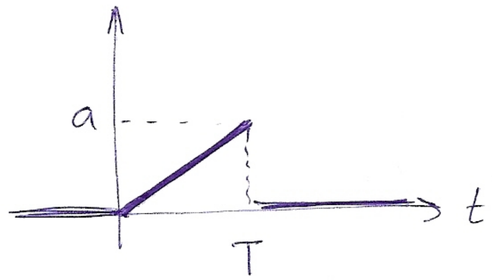
Exercise: Find Laplace transform of given periodic function with period T:



Let's define function $f_T(t)$



which can be written as

$$f_T(t) = \frac{a}{T} \cdot t \left(h(t) - h(t-T) \right)$$

$$\Rightarrow \quad \mathcal{L}\{f(t)\} = \frac{\mathcal{L}\{f_T(t)\}}{1 - e^{-sT}}$$

Let's try to compute $\mathcal{L}\{f_T(t)\}$ "by hand", first we need to write $f_T(t)$ in another form:

$$f_T(t) = \frac{a}{T} \cdot t \cdot h(t) - \frac{a}{T}(t - T + T)\, h(t-T) =$$

$$= \frac{a}{T}\, t\, h(t) - \frac{a}{T}(t-T)\, h(t-T) - a\, h(t-T)$$

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$$\Rightarrow \mathcal{L}\{f_T(t)\} = \frac{a}{T}\mathcal{L}\{t\} - e^{-sT}\mathcal{L}\{\frac{a}{T}t\} -$$

$$- a e^{-sT}\mathcal{L}\{h(t)\} = \underline{\frac{a}{T}\frac{1}{s^2} - e^{-sT}\frac{a}{T}\frac{1}{s^2} -}$$

$$\underline{- a e^{-sT}\frac{1}{s}}$$

or with Maple

```
restart;
with(inttrans):
assume(T>0):
f:= a/T*t*(Heaviside(t)-Heaviside(t-T));
laplace(f,t,s);
```

8) Convolution

<u>Definition:</u> The convolution of functions $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

<u>Remark:</u> Convolution has following properties

1) $f * g = g * f$ (commutative law)
2) $(f_1 + f_2) * g = f_1 * g + f_2 * g$

Laplace transform of convolution

$$\mathcal{L}\{f(t) * g(t)\} = \int_0^\infty \left(\int_0^t f(t-\tau)\, g(\tau)\, d\tau \right) e^{-st}\, dt$$

i.e. we have to compute the double integral over area:

$$\left.\begin{array}{l} 0 < \tau < t \\ 0 < t < \infty \end{array}\right\} \text{ we can change the order of variables to } \begin{array}{l} \tau < t < \infty \\ 0 < \tau < \infty \end{array}$$

So we have

$$\underline{\underline{\mathcal{L}\{f(t)*g(t)\}}} = \int_0^\infty \left(\int_\tau^\infty f(t-\tau)\, g(\tau)\, e^{-st}\, dt \right) d\tau =$$

$$= \left[\begin{array}{c} \text{substitute} \\ z = t-\tau \\ dz = dt \end{array} \right] = \int_0^\infty \left(\int_0^\infty f(z)\, g(\tau)\, e^{-s(z+\tau)}\, dz \right) d\tau =$$

$$= \int_0^\infty f(z)\, e^{-sz}\, dz \cdot \int_0^\infty g(\tau)\, e^{-s\tau}\, d\tau =$$

$$= \underline{\underline{\mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}}}$$

Remarks: Duhamel's integral

Consider we have linear ordinary differential equation of n-th order in form

$$x^{(n)}(t) + a_1 x^{(n-1)}(t) + \dots + a_{n-1} x'(t) + a_n x(t) = f(t)$$

with zero initial conditions

$$(\text{i.e. } x^{(n-1)}(0) = x^{(n-2)}(0) = \dots = x(0) = 0)$$

and $a_1, a_2, \dots, a_n$ are constants

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after Laplace transform

$$\left(s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n\right) X(s) = F(s)$$

where $X(s) = \mathcal{L}\{x(t)\}$

$F(s) = \mathcal{L}\{f(t)\}$

$\Rightarrow$ the Laplace transform of solution is

$$X(s) = \underbrace{\frac{1}{s^n + \dots + a_n}}_{= P(s)} \cdot F(s)$$

$= P(s)$ ... transfer function (in czech přenosová funkce)

$$X(s) = P(s)\,F(s)$$

$$\Rightarrow \quad x(t) = \mathcal{L}^{-1}\{P(s)\,F(s)\} = \int_0^t p(\tau)\,f(t-\tau)\,d\tau$$

where $p = \mathcal{L}^{-1}\{P(s)\}$

Now consider $f(t) = h(t)$ ($h(t)$ is Heaviside function). We know that $\mathcal{L}\{h(t)\} = \frac{1}{s}$

So the solution is $X_1(s) = P(s)\,\frac{1}{s}$

$\Rightarrow \quad P(s) = s \cdot X_1(s) \quad \Rightarrow$ the transfer function can be computed from known response to "unit input" (i.e. from known solution x_1 for $f(t) = h(t)$).

The solution for any input $f(t)$ is

$$X(s) = s\,X_1(s)\,F(s)$$

i.e.
$$x(t) = \mathcal{L}^{-1}\{s X_1(s)\} * f(t) =$$

$$= \dot{x}_1'(t) * f(t)$$

$$\boxed{x(t) = \int_0^t x_1'(\tau)\, f(t-\tau)\, d\tau}$$

<u>Duhamel's integral</u> (we first "measure" the response $x_1(t)$ on Heaviside function and after we can compute response $x(t)$ for any ~~xxx~~ "input" $f(t)$).

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Exercise: Use Duhamel's integral for following problem:

$$x''(t) + x(t) = f(t), \quad x(0) = x'(0) = 0$$

a) $f(t) = \sin t$

b) $f(t) = e^{t}$

c) $f(t) = e^{-t^2}$

---

consider $f(t) = h(t)$

$\Rightarrow$ after Laplace transform

$$s^2 X_1(s) + X_1(s) = \frac{1}{s}$$

$$X_1(s) = \frac{1}{s(s^2+1)}$$

$$x_1(t) = \mathcal{L}^{-1}\{X_1(s)\}$$

with Maple:

```
with(inttrans):
invlaplace(1/s/(s^2+1), s,t);
```

$2\sin^2\left(\frac{1}{2}t\right)$

or better

```
invlaplace(1/(s^3+s), s,t);
```

$1-\cos t$

i.e. $x_1(t) = 1 - \cos t$

$x_1'(t) = \sin t$

$\Longrightarrow$ any solution can be obtained like

$$x(t) = \int_0^t \sin \tau \cdot f(t-\tau)\, d\tau$$

So again with Maple:

a)
```
x := int( sin(u) * sin(t-u), u=0..t);
with(plots):
plot(x, t=0..3);
```

b)
```
x := int( sin(u) * exp(t-u), u=0..t);
```

—— " ——

c)
```
x := int( sin(u) * exp(-(t-u)^2), u=0..t);
```

G5

9) Dirac "function"

In physics we often use models of finite mass or charge located in single point, so the density goes to infinity. To describe this, English physicist P.A.M. Dirac introduced Dirac "function" $\delta(t)$ as

$$\left.\begin{aligned} &\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases} \\ &\text{and} \\ &\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \end{aligned}\right\}$$

this is inappropriate definition (integral should be zero according to the first property).

The definition of $\delta(t)$ was improved later to definition:

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \\ &\text{and} \\ &\int_{-\infty}^{\infty} f(t)\, \delta(t-\tau)\, dt = f(\tau) \end{aligned}\right\} \text{Ok}$$

Let's formally compute Laplace transform of $\eta'(t-\tau)$, $\tau > 0$ (η denotes Heaviside f.)

$$\mathcal{L}\{\eta'(t-\tau)\} = s \cdot \mathcal{L}\{\eta(t-\tau)\} =$$

$$= s \frac{e^{-s\tau}}{s} = e^{-s\tau}$$

and now using the ~~def~~ later definition of $\delta(t)$ we can write

$$e^{-s\tau} = \int_{\infty}^{\infty} \delta(t-\tau) e^{-st} dt = [\text{if } \tau > 0] =$$

$$= \int_{0}^{\infty} \delta(t-\tau) e^{-st} dt = \mathcal{L}\{\delta(t-\tau)\}$$

So we see, that $\mathcal{L}\{\eta'(t-\tau)\} = \mathcal{L}\{\delta(t-\tau)\}$

In generalized functions concept (theory of distributions) we really consider

$$\eta'(t-\tau) = \delta(t-\tau)$$

and $\mathcal{L}\{\delta(t-\tau)\} = e^{-s\tau}$

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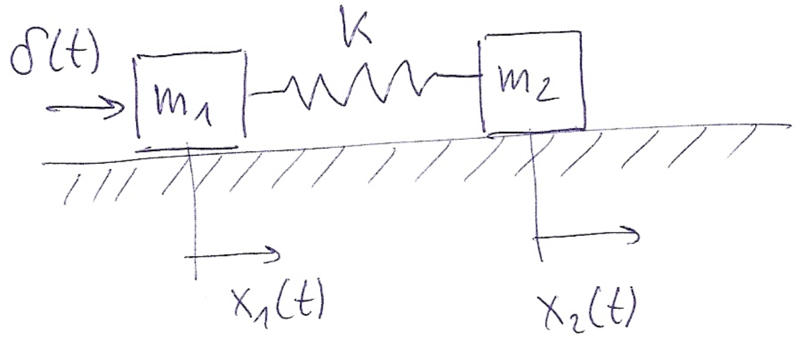
i.e. $\mathcal{L}\{\delta(t)\} = 1$

(in Maple $\delta(t)$ is denoted as Dirac (t))

Exercise : Consider two bodies connected by spring lying on surface without friction. One body is hit by hammer in $t=0$.



$$m_1 \ddot{x}_1(t) + K(x_1(t) - x_2(t)) = \delta(t)$$
$$m_2 \ddot{x}_2(t) - K(x_1(t) - x_2(t)) = 0$$

$$x_1(0) = \dot{x}_1(0) = x_2(0) = \dot{x}_2(0) = 0$$

Laplace transform of equations

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$$(m_1 s^2 + k)\, X_1(s) - k\, X_2(s) = 1$$

$$(m_2 s^2 + k)\, X_2(s) - k\, X_1(s) = 0$$

where $X_1(s) = \mathcal{L}\{x_1(t)\}$

$X_2(s) = \mathcal{L}\{x_2(t)\}$

$$\begin{pmatrix} m_1 s^2 + k & -k \\ -k & m_2 s^2 + k \end{pmatrix} \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

see the Maple file

```
> m1:=1;m2:=10;k:=100;
```

$$m1 := 1$$

$$m2 := 10$$

$$k := 100$$

```
> res:=solve({(m1*s*s+k)*X1-k*X2=1,-k*X1+(m2*s*s+k)*X2=0},{X1,X2});
```

$$res := \left\{ X1 = \frac{s^2+10}{s^2\,(s^2+110)}, X2 = \frac{10}{s^2\,(s^2+110)} \right\}$$

```
> X1(s):=subs(res,X1);
```

$$X1(s) := \frac{s^2+10}{s^2\,(s^2+110)}$$

```
> X2(s):=subs(res,X2);
```

$$X2(s) := \frac{10}{s^2\,(s^2+110)}$$

```
> with(inttrans):x1(t):=invlaplace(X1(s),s,t);
```

$$x1(t) := \frac{1}{11}\,t + \frac{1}{121}\,\sqrt{110}\,\sin(\sqrt{110}\,t)$$

```
> x2(t):=invlaplace(X2(s),s,t);
```

$$x2(t) := \frac{1}{11}\,t - \frac{1}{1210}\,\sqrt{110}\,\sin(\sqrt{110}\,t)$$

```
> with(plots):
> p1:=plot(x1(t),t=0..10,color=red):
> p2:=plot(x2(t),t=0..10,color=blue):
> display({p1,p2},axes=boxed);
```

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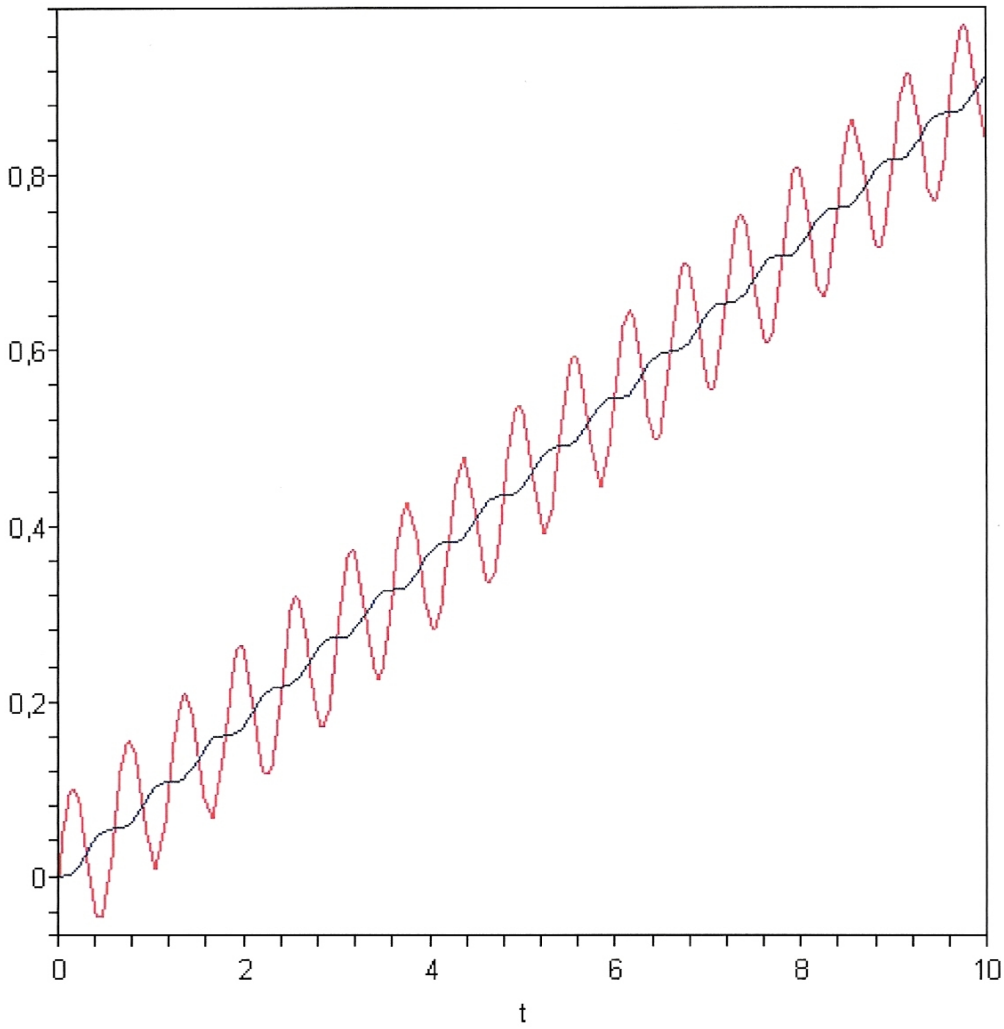

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