Remark: Heaviside function

$$h(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

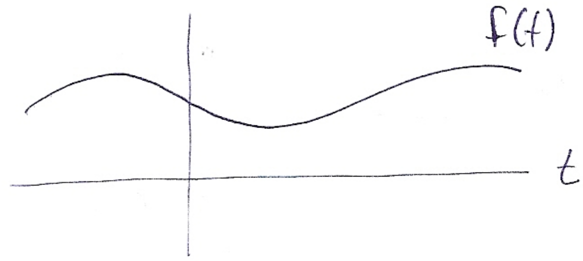
the Laplace transform of $h(t)$ is

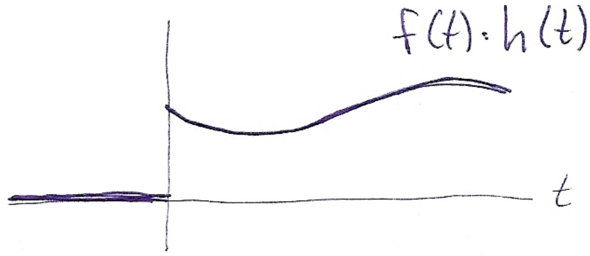
$$\mathcal{L}\{h(t)\} = \mathcal{L}\{e^{0 \cdot t}\} = \frac{1}{s}$$

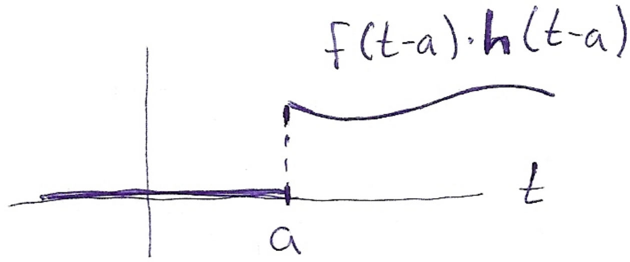
in Maple: laplace (Heaviside (t), t, s);

2) Laplace transform of function f(t) shifted in argument t

consider graph of f(t):







36

we want to compute the Laplace transform of $f(t-a)\cdot h(t-a)$, where h is Heaviside function

$$\mathcal{L}\{f(t-a)\,h(t-a)\} = \int_0^\infty f(t-a)\,h(t-a)\,e^{-st}\,dt =$$

$$= \int_a^\infty f(t-a)\,e^{-st}\,dt = \begin{bmatrix} \text{substitution} \\ u = t-a \\ du = dt \end{bmatrix} =$$

$$= \int_0^\infty f(u)\,e^{-s(u+a)}\,du = e^{-sa}\int_0^\infty f(u)\,e^{-su}\,du =$$

$$= \underline{\underline{e^{-sa}\,F(s)}} \qquad \left(\text{where } F(s) \equiv \mathcal{L}\{f(t)\}\right)$$

<u>Example:</u>

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

$$\mathcal{L}\{\sin(t-\tfrac{\pi}{2})\} = e^{-\frac{\pi}{2}s}\cdot\frac{1}{s^2+1}$$

3) <u>shift of argument in transformed function</u>

following property holds

$$\mathcal{L}\{e^{at}\cdot f(t)\} = \int_0^\infty e^{at} f(t)\,e^{-st}\,dt =$$

$$= \int_0^\infty f(t)\,e^{-t(s-a)}\,dt = \underline{\underline{F(s-a)}},$$

where $F(s) = \mathcal{L}\{f(t)\}$.

<u>Example:</u> The last property can be used for inverse transform, i.e. we know

$$F(s) = \mathcal{L}\{f(t)\} = \frac{1}{s^2+2s+2}$$

(which can represent a Laplace transform of some transfer function, in czech přenosové funkce)

and we need $f(t) = ?$

we can write $F(s) = \dfrac{1}{s^2+2s+2} = \dfrac{1}{(s+1)^2+1}$

and we know $\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$

and that $\mathcal{L}\{e^{-t} \cdot \sin t\} = \dfrac{1}{(s+1)^2+1}$

$$\Rightarrow \quad \underline{f(t) = e^{-t}\sin t}$$

4) Laplace transform of derivative $f'(t)$

let's consider a continuous function $f(t)$
(i.e. $f'(t)$ is bounded and therefore integrable)

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)\, e^{-st}\, dt =$$

$$= \left[\begin{array}{ll} \text{per-parts} & \\ u' = f' & v = e^{-st} \\ u = f & v' = -s e^{-st} \end{array}\right] = \left[f(t)\, e^{-st}\right]_0^\infty -$$

$$- \int_0^\infty f(t)\,(-s)\, e^{-st}\, dt = -f(0) +$$

$$+ s \int_0^\infty f(t) e^{-st}\, dt = \underline{\underline{-f(0) + s\, F(s)}} ,$$

where $F(s) = \mathcal{L}\{f(t)\}$

Laplace transform of n-th derivative $f^{(n)}(t)$ can be computed by recursion :

(39)

$$\mathcal{L}\{f^{(n)}(t)\} = s\mathcal{L}\{f^{(n-1)}(t)\} - f^{(n-1)}(0) =$$

$$= s^2\mathcal{L}\{f^{(n-2)}(t)\} - s f^{(n-2)}(0) - f^{(n-1)}(0) =$$

$$= \cdots\cdots = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) -$$

$$- s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{(n-1)}(0) =$$

$$= s^n F(s) - \sum_{j=1}^{n} s^{n-j} f^{(j-1)}(0)$$

Remark: Note, that the derivatives $f^{(j-1)}(0)$ are the derivatives from the right side.

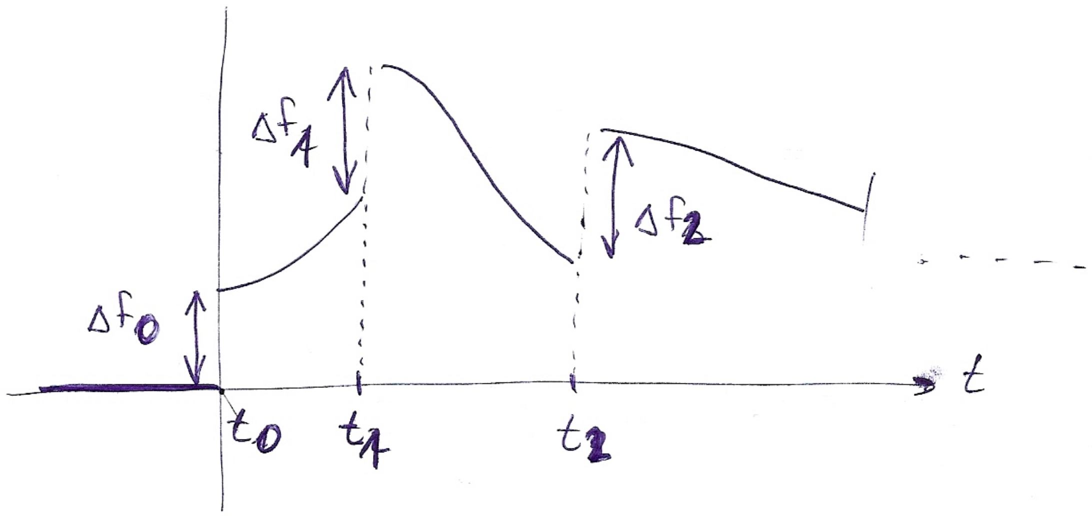
Remark: Laplace transform of $f'(t)$ for the case of piecewise continuous function $f(t)$.

Let's consider a function $f(t)$ with finite number (or ~~maximally~~ at the most countable) of discontinuities in $t = t_K$. Let $f(t), f'(t) \in T_\alpha$. Let's denote

(40)

$$\Delta f_K = f(t_K^+) - f(t_K^-)$$

see example



then

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}\,dt =$$

$$= \sum_{K=1}^{\infty} \int_{t_{K-1}}^{t_K} f'(t)\,e^{-st}\,dt = \begin{bmatrix} \text{per parts} & \\ u' = f' & v = e^{-st} \\ u = f & v' = -s e^{-st} \end{bmatrix} =$$

$$= \sum_{K=1}^{\infty} \left(\left[f(t)e^{-st}\right]_{t_{K-1}}^{t_K} + s\int_{t_{K-1}}^{t_K} f(t)\,e^{-st}\,dt \right) =$$

$$= \sum_{K=1}^{\infty} \left[f(t)e^{-st}\right]_{t_{K-1}}^{t_K} + s\int_0^\infty f(t)\,e^{-st}\,dt =$$

$$= s\cdot\mathcal{L}\{f(t)\} - f(0+) - \sum_{K=1}^{\infty} \Delta f_K\, e^{-st_K}$$

41

Exercise: Solve $\dot{y} - 3y = t$, $y(0) = 1$ by Laplace transform. $\left(\dot{y} = \frac{dy}{dt} \right)$

$$\dot{y}(t) - 3y(t) = t \quad \Big| \mathcal{L}$$

$$\mathcal{L}\{\dot{y}(t)\} - 3\mathcal{L}\{y(t)\} = \mathcal{L}\{t\}$$

$$s\mathcal{L}\{y(t)\} - y(0) - 3\mathcal{L}\{y(t)\} = \mathcal{L}\{t\}$$

using Maple laplace(t, t, s); $\rightarrow \mathcal{L}\{t\} = \frac{1}{s^2}$

let's denote $Y(s) = \mathcal{L}\{y(t)\}$

$$sY(s) - 1 - 3Y(s) = \frac{1}{s^2}$$

$$(s-3)Y(s) = \frac{1}{s^2} + 1$$

$$Y(s) = \frac{s^2+1}{s^2(s-3)}$$

? how to perform inverse transform

- using dictionary of Laplace transform

42

we can write $Y(s)$ as a sum of partial fractions

$$Y(s) = \frac{s^2+1}{s^2(s-3)} = -\frac{1}{3}\frac{1}{s^2} - \frac{1}{9}\frac{1}{s} + \frac{10}{9}\frac{1}{s-3}$$

(by hands or by Maple
convert ((s*s+1)/s/s/(s-3), parfrac, s);)

from dictionary: $\mathcal{L}\{t\} = \frac{1}{s^2}$

$$\mathcal{L}\{h(t)\} = \frac{1}{s}$$

$$\mathcal{L}\{e^{at} \cdot f(t)\} = F(s-a)$$

$$\Rightarrow y(t) = -\frac{1}{3} \cdot t - \frac{1}{9} h(t) + \frac{10}{9} e^{3t} \cdot h(t)$$

considering $t \geq 0$

$$y(t) = -\frac{t}{3} - \frac{1}{9} + \frac{10}{9} e^{3t}$$

- using Maple directly

  invlaplace((s*s+1)/s/s/(s-3), s, t);

- there exists also the formula for inverse transform.

5) inverse Laplace transform

The Laplace transform is not one-to-one mapping, there exists an infinite number of functions, which have the same Laplace transform. Fortunately they differ only in discontinuities. If we permit only functions $f(t)$, which

a) in discontinuity $t = t_i$ satisfy

$$f(t_i) = \left(\lim_{t \to t_i -} f(t) + \lim_{t \to t_i +} f(t) \right) / 2$$

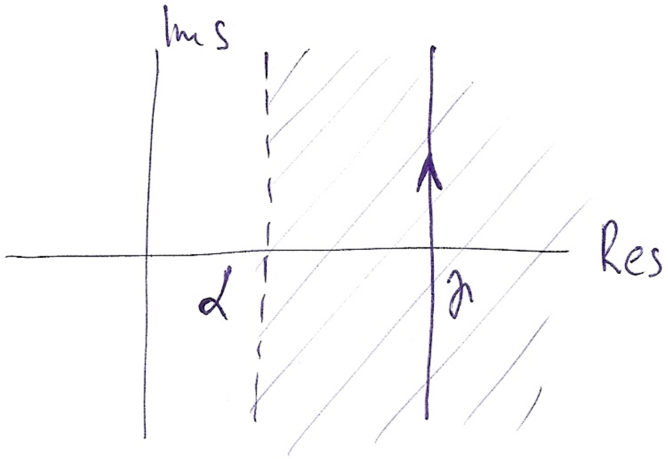
b) $f(t) = 0$ for $t < 0$,

then the mapping $f(t) \to F(s)$ is the one-to-one mapping (in czech prosté zobrazení) and inverse mapping exists.

Theorem: Let $f(t)$ be the original function and $F(s)$ its Laplace transform defined for $\mathrm{Re}\, s > \alpha$ $(\alpha \geq 0)$, then

(44)

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s) e^{st} \, ds = \mathcal{L}^{-1}\{F(s)\}$$



$\alpha, \gamma \in \mathbb{R}$, $\gamma > \alpha$

Remark: The inverse Laplace transform formula can be derived using Fourier transform formula (which we will derive later):

Fourier transform $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$

inverse Fourier transform $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} dt$

The Laplace transform of $f(t)$ is

$F(s) = \int_0^{\infty} f(t) e^{-st}$ and

(45)

The Fourier transform of $e^{-\gamma t} f(t)$ is

$$\int_{-\infty}^{\infty} e^{-\gamma t} f(t)\, e^{-i\omega t}\, dt = \left[f(t) = 0 \text{ for } t < 0 \right] =$$

$$= \int_{0}^{\infty} f(t)\, e^{-(\gamma + i\omega)t}\, dt = F(\gamma + i\omega),$$

which can be seen as Laplace transform of $f(t)$, where "s" is replaced by "$\gamma + i\omega$".

Now the inverse Fourier transform gives

$$e^{-\gamma t} f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\gamma + i\omega)\, e^{i\omega t}\, d\omega$$

$$\underline{f(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\gamma + i\omega)\, e^{(\gamma + i\omega)t}\, d\omega =$$

$$= \left[\begin{array}{l} \text{subst.} \\ s = \gamma + i\omega \\ ds = i \cdot d\omega \end{array} \right] = \underline{\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s)\, e^{st}\, ds}$$

$\square$

<u>Remark:</u> The integral in the formula

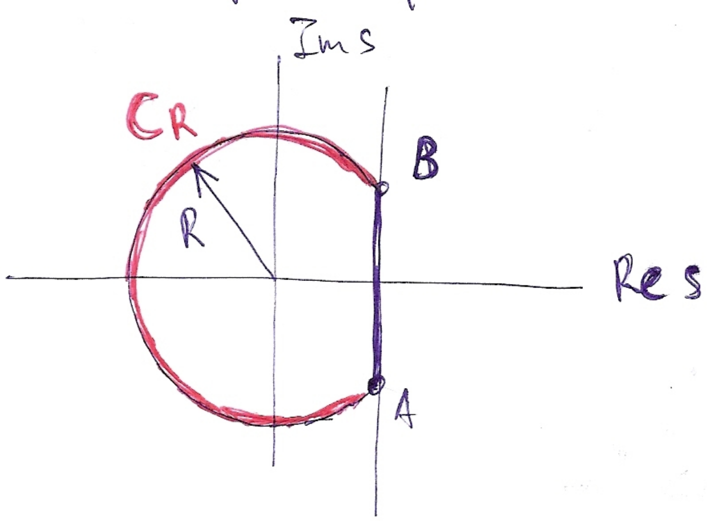
$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s) e^{st} \, ds$$

can be computed using residue theorem (p. 27). It is possible to show, that $F(s)$ is analytic function for $\mathrm{Re} > \alpha$ and that $F(s) \to 0$ for $s \to \infty$.

We further need <u>Jordan's lemma</u>:

$$\int_{C_R} F(s) e^{st} \, ds \to 0 \quad \text{if } R \to \infty$$

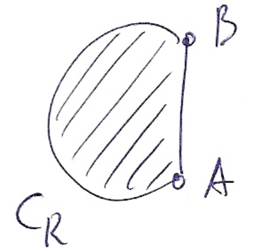
($F(s)$ must be analytic on C_R, i.e. we have to avoid singular points)



Therefore for $R \to \infty$

$$\int_{\overline{AB}} F(s)e^{st}\,ds + \int_{C_R} F(s)e^{st}\,ds \to \int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}\,ds$$

and all singular points of $F(s)$ are inside $\overline{AB} \cup C_R$



$\Rightarrow$ according to residue theorem

$$\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}\,ds = 2\pi i \sum_K \operatorname{Res}\left[F(s)e^{st}, s_K\right]$$

$$\Rightarrow \boxed{f(t) = \sum_k \operatorname{Res}\left[F(s)e^{st}, s_k\right]}$$

where s_k are all singular points of $F(s)$

(48)

→ by Maple:

```
restart:
readlib(singular):
readlib(residue):
F:=(s*s+1)/s/s/(s-3);
singular(F);      → s=0, s=3
r1:=residue(F*exp(s*t), s=0);
r2:=residue(F*exp(s*t), s=3);
f:= r1+r2;
```

49

6) Laplace transform of integral

$$\mathcal{L}\left\{ \int_0^t f(\tau)\, d\tau \right\} = \frac{F(s)}{s},$$

where $F(s) = \mathcal{L}\{f(t)\}$

<u>Proofs</u> let's define $g(t) = \int_0^t f(\tau) d\tau$

$\Rightarrow$ $g'(t) = f(t)$ and $g(0) = 0$.

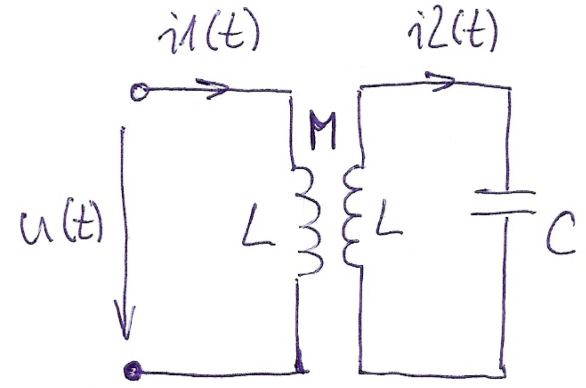
It is possible to show $g(t) \in T_\mathcal{L}$.

$$\underline{\mathcal{L}\{f(t)\}} = \mathcal{L}\{g'(t)\} = \underline{s\,\mathcal{L}\{g(t)\}} - \underbrace{\cancel{g(0)}}_{=0}$$

$$\Rightarrow \mathcal{L}\{g(t)\} = \frac{\mathcal{L}\{f(t)\}}{s} \qquad \square$$

50

<u>Exercise:</u> It is given following circuit



initial conditions: $i1(0) = 0$, $i2(0) = 0$

constants: inductance L, M
capacitance C

forcing term: $u(t) = h(t)$ ... Heaviside funct.

The circuit is described by the set of equation:

$$\mathcal{L} \left| \begin{aligned} L\, i1'(t) + M\, i2'(t) &= \overset{h(t)}{u(t)} \\ M\, i1'(t) + L\, i2'(t) + \frac{1}{C}\int_0^t i2(\tau)\,d\tau &= 0 \end{aligned} \right.$$

$$\begin{aligned} Ls\, I1(s) + Ms\, I2(s) &= \frac{1}{s} \\ Ms\, I1(s) + Ls\, I2(s) + \frac{1}{Cs} I2(s) &= 0 \end{aligned} \qquad (*)$$

---

where $I1(s) = \mathcal{L}\{i1(t)\}$, ...

we need to solve $(*)$, see Maple sheet

> restart:

> res:=solve({L*s*i1+M*s*i2=1/s,M*s*i1+(L*s+1/C/s)*i2=0},{i1,i2});

$$res := \left\{ i2 = -\frac{M\,C}{L^2\,s^2\,C+L-M^2\,C\,s^2},\, i1 = \frac{L\,s^2\,C+1}{s^2\,(L^2\,s^2\,C+L-M^2\,C\,s^2)} \right\}$$

> i1(s):=subs(res,i1);

$$i1(s) := \frac{L\,s^2\,C+1}{s^2\,(L^2\,s^2\,C+L-M^2\,C\,s^2)}$$

> i2(s):=subs(res,i2);

$$i2(s) := -\frac{M\,C}{L^2\,s^2\,C+L-M^2\,C\,s^2}$$

> with(inttrans):i1(t):=invlaplace(i1(s),s,t);

$$i1(t) := \frac{t}{L} + \frac{\sinh\left(\frac{\sqrt{L\,C\,(M^2-L^2)}\,t}{C\,(L^2-M^2)}\right)\sqrt{L\,C\,(M^2-L^2)}\,M^2}{(M^2-L^2)\,L^2}$$

> i2(t):=invlaplace(i2(s),s,t);

$$i2(t) := \frac{\sinh\left(\frac{\sqrt{L\,C\,(M^2-L^2)}\,t}{C\,(L^2-M^2)}\right)\sqrt{L\,C\,(M^2-L^2)}\,M}{(L^2-M^2)\,L}$$

> with(plots):I1(t):=subs(C=1,M=1,L=2,i1(t));

Warning, the name changecoords has been redefined

$$I1(t) := \frac{1}{2}\,t - \frac{1}{12}\sinh\left(\frac{1}{3}\sqrt{-6}\,t\right)\sqrt{-6}$$

> I2(t):=subs(C=1,M=1,L=2,i2(t));

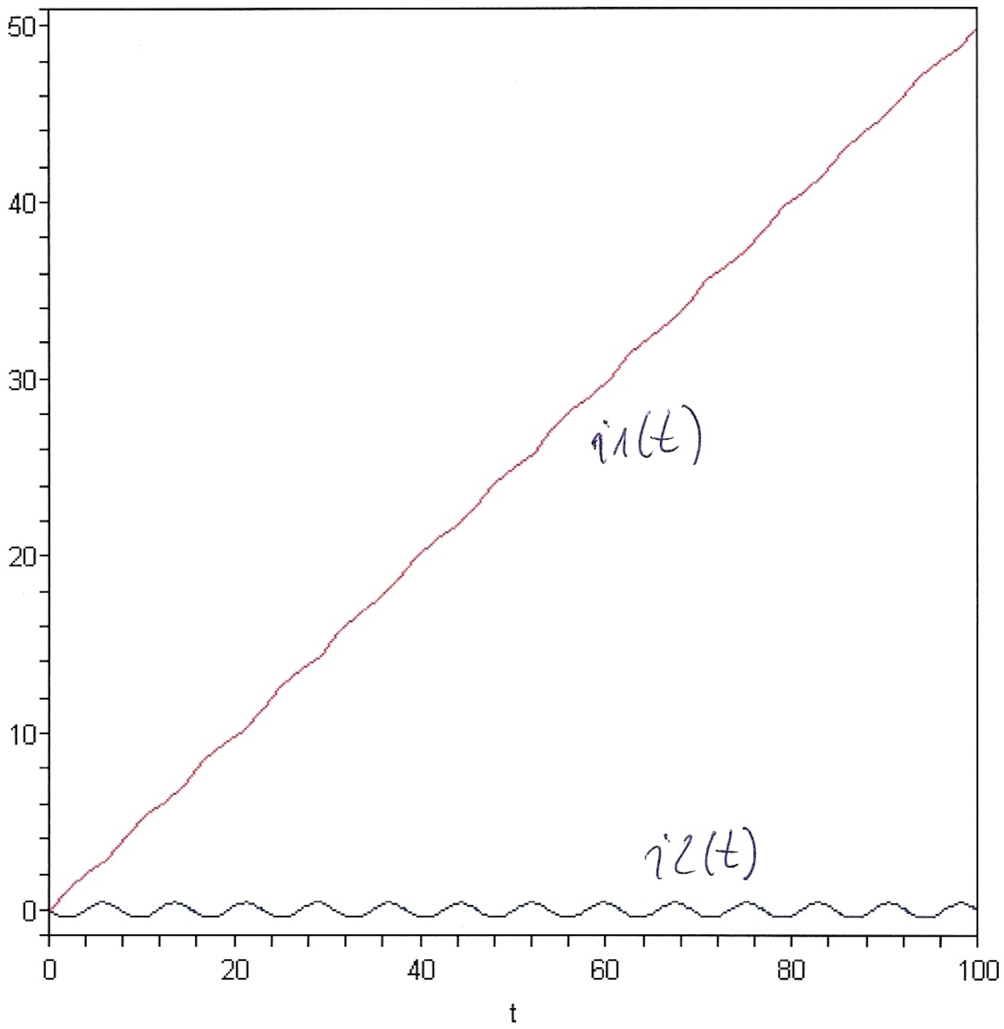
$$I2(t) := \frac{1}{6}\sinh\left(\frac{1}{3}\sqrt{-6}\,t\right)\sqrt{-6}$$

> p1:=plot(I1(t),t=0..100,color=red):

> p2:=plot(I2(t),t=0..100,color=blue):

> display({p1,p2},axes=boxed);

52



>

<u>Remark:</u> $i1(t) \to \infty$ for $t \to \infty$

because the loop on the left has zero resistance.