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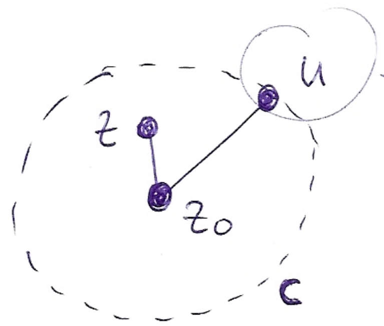

value along curve c

then $$f(z) = \frac{1}{2\pi i} \oint_C f(u)\, \underbrace{a\left(1+q+q^2+q^3+\cdots\right)}_{\text{converging for } |z-z_0|<R} du$$

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(u)}{u-z_0} + \frac{1}{2\pi i} \oint \frac{f(u)}{(u-z_0)^2} du \cdot (z-z_0) +$$

$$+ \cdots + \frac{1}{2\pi i} \oint_C \frac{f(u)}{(u-z)^{k+1}} du \cdot (z-z_0)^k + \cdots$$

so we can write

$$f(z) = \sum_{k=0}^{\infty} c_k (z-z_0)^k$$

with

$$c_k = \frac{1}{2\pi i} \oint_C \frac{f(u)}{(u-z_0)^{k+1}} du = \frac{f^{(k)}(z_0)}{k!}$$

Ex: Write the Taylor expansion for $f(z) = \sin z$ in point $z_0 = 0$.

The function $\sin z$ is analytic everywhere (for any complex z) $\Rightarrow$ Taylor expansion should converge also for any z

$f(0) = 0$

$f'(0) = \cos(0) = 1$

$f''(0) = -\sin(0) = 0$

$f'''(0) = -\cos(0) = -1$

$\vdots$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

Ex: Write the Taylor expansion for $f(z) = \dfrac{\sin z}{z^2+1}$ in $z_0 = 0$.

we need to know, where is $f(z)$ analytic.

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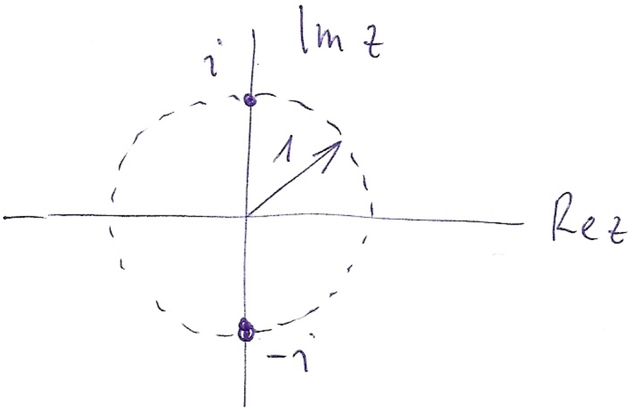
with Maple:

```
restart:
readlib(singular):
f:= sin(z)/(z^2+1);
singular(f, z);
```

$\longrightarrow \pm i$



$\Rightarrow$ Taylor expansion converges for $|z| < 1$

```
taylor(f, z=0, 10);
```

or

```
evalf(taylor( — , — ));
```

Another example: Taylor expansion of $f(z) = \operatorname{tg} z$

in $z_0 = 0$ and $z_0 = i$.

```
restart:
readlib(singular):
f:= tan(z);
```

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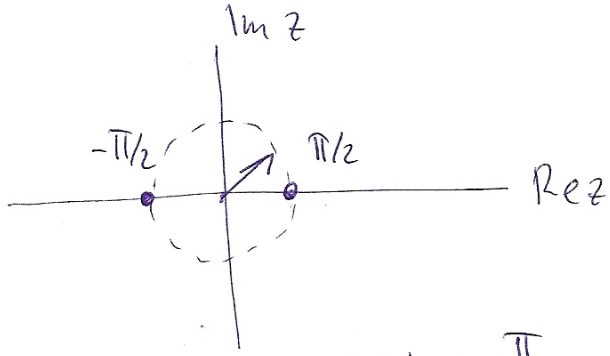
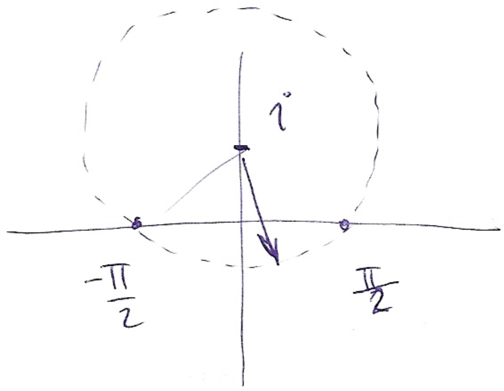
singular(f, z); $\rightarrow \frac{\pi}{2} + k\pi$

for $z_0 = 0$



converges for $|z| < \frac{\pi}{2}$

for $z_0 = i$



convergence for $|z - i| < \sqrt{1 + \frac{\pi^2}{4}}$

taylor(. . . .

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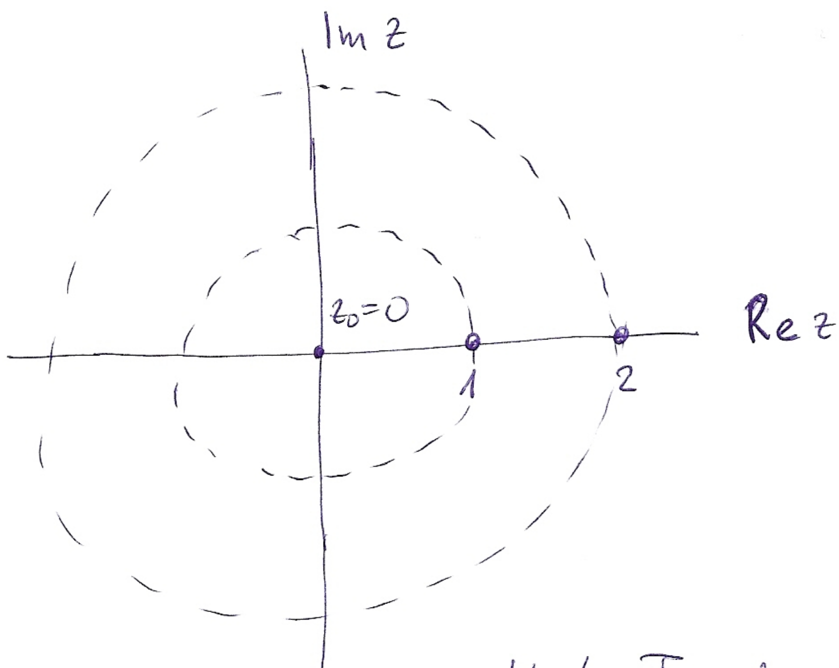
Laurent series

Let's start with another example of Taylor expansion of $f(z) = \dfrac{1}{z^2 - 3z + 2}$ in $z_0 = 0$.

Singular points are $z = 1$ and $z = 2$

since $$\frac{1}{z^2 - 3z + 2} = \frac{1}{z-2} - \frac{1}{z-1}$$

(by Maple
```
readlib(singular):
f := 1/(z^2 - 3*z + 2);
singular(f, z);
```
)



a) we already know, that Taylor expansion exist for $|z| < 1$

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b) if we need an expansion of $f(z)$ for $|z|>1$, we do following:

$$f(z) = \frac{1}{z^2-3z+2} = \frac{1}{z-2} - \frac{1}{z-1}$$

where

$$\frac{1}{z-2} = -\frac{1}{2}\,\frac{1}{1-\underbrace{\frac{z}{2}}_{=q}} = -\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k$$

which converges for $\left|\frac{z}{2}\right|<1$, i.e. $|z|<2$

and

$$-\frac{1}{z-1} = -\frac{1}{z}\,\frac{1}{1-\underbrace{\frac{1}{z}}_{=q}} = -\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k$$

which converges for $\left|\frac{1}{z}\right|<$, i.e. $|z|>1$

So finally

$$f(z) = \underbrace{\underbrace{\cdots - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z}}_{\text{main part}} \underbrace{- \frac{1}{2} - \frac{z}{2^2} - \frac{z^2}{2^3} - \cdots}_{\text{analytic part}}}_{\text{Laurent expansion}}$$

Laurent expansion, which is convergent for $1<|z|<2$

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c) it is possible to construct the Laurent expansion also for $|z|>2$.

we already know that

$$-\frac{1}{z-1} = -\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k \quad \text{for} \quad |z|>1$$

and we can write

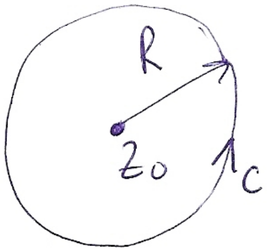
$$\frac{1}{z-2} = \frac{1}{z}\,\frac{1}{1-\frac{2}{z}} = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^k$$

$$\text{for} \quad \left|\frac{2}{z}\right|<1, \quad \text{i.e.} \quad |z|>2$$

$$\Rightarrow \underline{f(z) = -\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k + \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^k},$$

$\underline{\text{for } |z|>2}$ ... this expansion doesn't have analytic part

Theorem: Let c be a curve with the shape of circle with center in z_0.



Consider $f(z)$ analytic in domain $G: 0 < |z - z_0| < R$ with isolated singular point in $z = z_0$. Then Laurent ~~expansi~~ serie $f(z) = \sum_{k=-\infty}^{\infty} c_k (z-z_0)^k$ converges in G.

Exercise: Find all singular point of function

$$f(z) = \frac{1}{z^3 - 2z^2 - 4}$$

and write a Laurent expansions in these points.

```
restart;
readlib(singular):
f:= 1/(z^3+3*z^2-4);
singular(f, z);  →  1, -2
```

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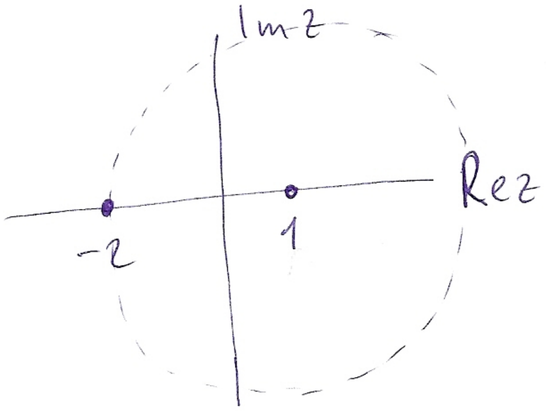
Laurent expansion in $z=1$



convergence for

$0<|z-1|<3$

series(f, z=1, 8);

or with(numapprox):

laurent(f, z=1, 8);

Laurent expansion in $z=-2$

convergence for $0<|z+2|<3$

series(f, z=-2, 8);

Calculus of Residues

The Laurent series of function $f(z)$, which is analytic in region $0 < |z - z_0| < R$ has the form $f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$,

where

$$a_k = \frac{1}{2\pi i} \int_{C_r} \frac{f(u)}{(u - z_0)^{k+1}} \, du$$

and c_r is the circle with radius r $(0 < r < R)$ and center z_0.

The coefficient $a_{-1} = \text{Res}[f(z), z = z_0]$ is called residue of $f(z)$ at the point z_0.

Example: compute $\text{Rez}[f, z=1]$ and $\text{Rez}[f, z=-2]$ for $f(z) = \dfrac{1}{z^3 - 3z^2 - 4}$

by Maple:

```
f := ------
readlib (residue):
residue (f, z=1);
```

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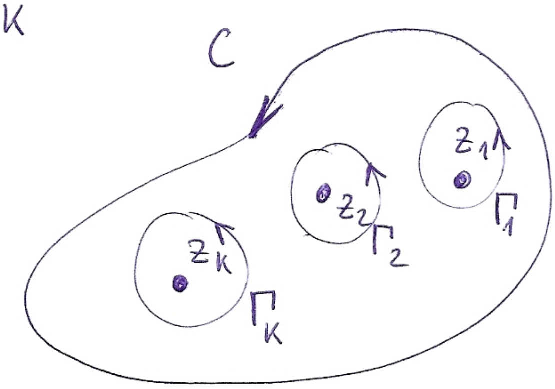
Residue theorem

Suppose that function $f(z)$ is analytic on curve c and in $\operatorname{int} c$ with exception of K isolated singular points $z_1, z_2, \ldots, z_K$



Applying the Cauchy theorem we can write

$$\oint_c f(z)\,dz = \oint_{\Gamma_1} f(z)\,dz + \oint_{\Gamma_2} f(z)\,dz + \cdots + \oint_{\Gamma_K} f(z)\,dz \ \Big| \cdot \frac{1}{2\pi i}$$

$$\oint_c f(z)\,dz \cdot \frac{1}{2\pi i} = \operatorname{Res}[f, z=z_1] + \operatorname{Res}[f, z=z_2] + \cdots + \operatorname{Res}[f, z=z_K]$$

i.e.

$$\oint_c f(z)\,dz = 2\pi i \sum_{j=1}^{K} \operatorname{Res}[f(z), z=z_j]$$

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<u>Rem:</u> Residue theorem is useful e.g. for evaluation of integrals.

<u>Example:</u> Compute $\oint_C \frac{dz}{z^3+4z}$, where

C is the curve : a) $|z|=2$

b) $|z|=1$

c) $|z|=3$

with possitive orientation.

Function $\frac{1}{z^3+4z} = \frac{1}{z(z^2+4)}$ has singular points $0, \pm 2i$. Let's evaluate residues by Maple

```
restart;
readlib(residue):
f:=1/(z^3+4*z);
residue(f, z=0);       → 1/4 = r1
residue(f, z=2*I);     → -1/8 = r2
residue(f, z=-2*I);    → -1/8 = r3
```

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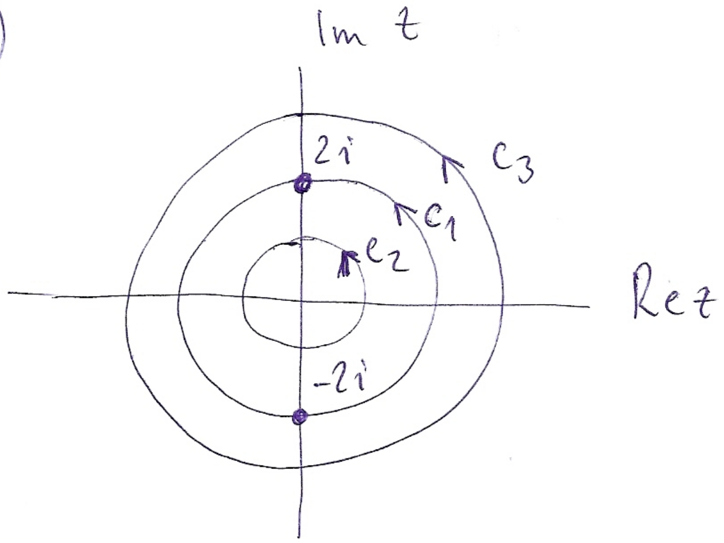

$\oint_{C_1} f(z)\,dz$ doesn't exist, since $f(z)$ is not analytic on C_1

$$\oint_{C_2} f(z)\,dz = 2\pi i \cdot r_1 = \frac{\pi i}{2}$$

$$\oint_{C_3} f(z)\,dz = 2\pi i\,(r_1 + r_2 + r_3) = 0.$$

INTEGRAL TRANSFORM

The motivation for integral transforms came from effort to transform differential equations into algebraic one to solve them easily. However problems can appear during inverse transform back to the original solution.

The general integral transform can be written as

$$F(s) = \int_a^b K(s,t)\, f(t)\, dt \,, \quad \text{where}$$

$f(t)$ is original (in czech: originál, vzor)
$F(s)$ is transform (in czech: obraz)
$K(s,t)$ is the kernel of transform (in czech: jádro transformace)

Laplace transform

is defined with $a=0$, $b=\infty$, $K(s,t)=e^{-st}$

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i.e. $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$

we consider $f(t) = 0$ for $t < 0$, s is complex, t is real

Which function $f(t)$ has the Laplace transform?

We define the class T_α of functions $f(t)$ which are

1) piecewise continuous in $(0; +\infty)$, i.e. they are bounded and not-defined in finite number of points

2) exists such $\alpha \in \mathbb{R}$, $M \geq 0$ that for all $t \in (0; \infty)$ holds

$$|f(t)| \leq M e^{\alpha t}$$

Rem. e.g. function $g(t) = e^{t^2} \notin T_\alpha$

(it growts faster then exponential function)

because t^2 grows faster than αt

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Existence of Laplace transform of $f(t)$

We consider $f(t) \in T_\alpha$, then

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st} dt = \lim_{T\to\infty} \int_0^T f(t) e^{-st} dt$$

let's try to find upper bound for

$$\int_0^T f(t) e^{-st} dt \leq \int_0^T |f(t) e^{-st}| dt =$$

$$= \int_0^T |f(t)| \cdot |e^{-st}| dt \leq \int_0^T M e^{\alpha t} |e^{-st}| dt =$$

$$= \int_0^T M e^{\alpha t} e^{-\mathrm{Re}\, s \cdot t} dt = \int_0^T M e^{(\alpha - \mathrm{Re}\, s)t} dt =$$

$$= \left[\frac{M}{\alpha - \mathrm{Re}\, s} e^{(\alpha - \mathrm{Re}\, s)} \right]_0^T = \frac{M}{\alpha - \mathrm{Re}\, s} \left(e^{(\alpha - \mathrm{Re}\, s)T} - 1 \right)$$

$$\mathcal{L}\{f(t)\} = \underbrace{\lim_{T\to\infty} \frac{M}{\alpha - \mathrm{Re}\, s} \left(e^{(\alpha - \mathrm{Re}\, s)T} - 1 \right)}$$

this limit has to be finite,
therefore $\underline{\alpha - \mathrm{Re}\, s < 0}$

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Following theorem holds:

If $f(t) \in T_\alpha$, then the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ is analytic function for $\mathrm{Re}\, s > \alpha$.

Exercise: Find $\mathcal{L}\{f(t)\}$ for $f(t) = e^{at}$, $a \in \mathbb{R}$

we start from definition

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at} e^{-st}\, dt = \int_0^\infty e^{(a-s)t}\, dt =$$

$$= \lim_{T\to\infty} \int_0^T e^{(a-s)t}\, dt = \lim_{T\to\infty} \left(\frac{1}{a-s}\left[e^{(a-s)t}\right]_0^T\right) =$$

$$= \frac{1}{a-s} \lim_{T\to\infty} \left(e^{(a-s)T} - 1\right) = \frac{1}{s-a}, \quad \mathrm{Re}\, s > a$$

(generally $s, a \in \mathbb{C}$, $\mathrm{Re}\, s > \mathrm{Re}\, a$)

by Maple:

```
restart:
with(inttrans):
laplace(exp(3*t),t,s);
```

Properties of Laplace transform

1) Linearity:

$$\mathcal{L}\{a f(t) + b g(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\},$$

where a, b are constants (generally complex).

Exercise: $\mathcal{L}\{\sin t\} = ?$

we can write $\sin t = \frac{1}{2i}(e^{it} - e^{-it})$

$$\Rightarrow \mathcal{L}\{\sin t\} = \frac{1}{2i}\left(\mathcal{L}\{e^{it}\} - \mathcal{L}\{e^{-it}\}\right) =$$

$$= \frac{1}{2i}\left(\frac{1}{s-i} - \frac{1}{s+i}\right) = \frac{1}{2i}\,\frac{2i}{s^2+1} =$$

$$= \frac{1}{s^2+1}, \quad \operatorname{Re} s > 0$$