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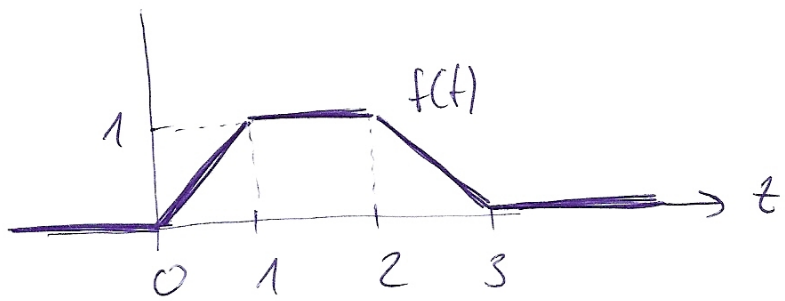
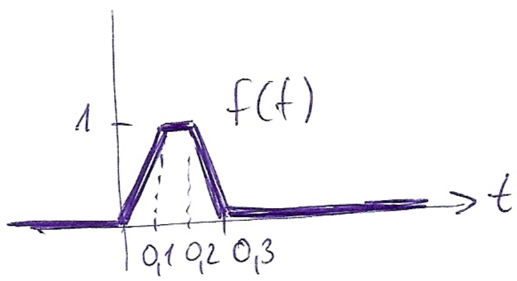
<u>Remark:</u> Let's consider some signal $f(t)$.
$|F(\omega)|$, where $F(\omega) = \mathcal{F}\{f(t)\}$, is proportional to so called spectral density and $|F(\omega)|^2$ to energy spectra. The integral $\int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega$ is proportional to total energy of signal.

<u>Exercise:</u> Find the energy spectra for signals

a)



b)



164 a)

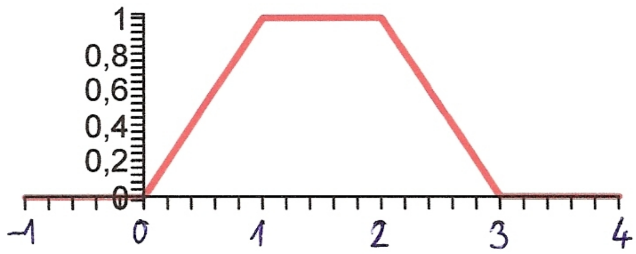
```
> restart:with(plots):
```

Warning, the name changecoords has been redefined

```
> f:=piecewise(x<0,0,x>0 and x<1, x, x>1 and x<2, 1, x>2 and x<3,
  3-x,x>3,0);
```

$$f := \begin{cases} 0 & x < 0 \\ x & 0 < x \textbf{ and } x < 1 \\ 1 & 1 < x \textbf{ and } x < 2 \\ 3 - x & 2 < x \textbf{ and } x < 3 \\ 0 & 3 < x \end{cases}$$

```
> plot(f,x=-1..4);
```



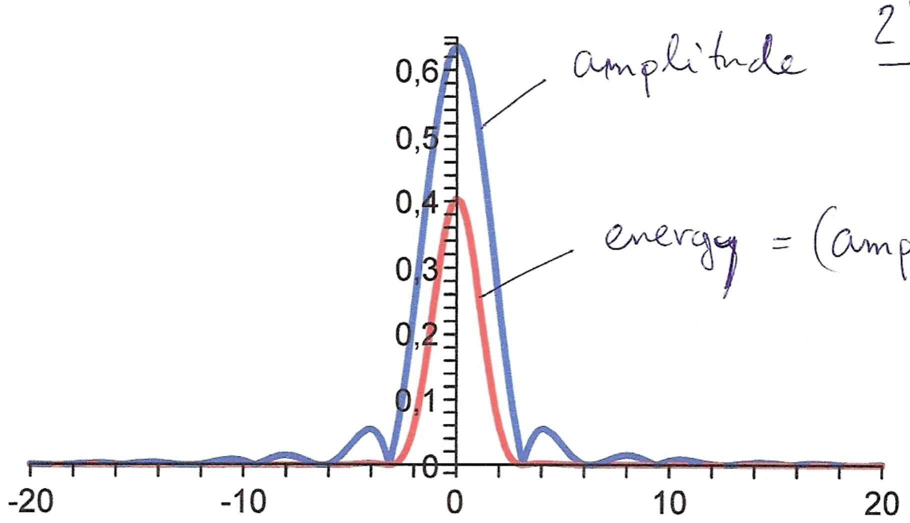

```
> F(w):=int(f*exp(-I*x*w),x=0..3);
```

$$F(w) := -\frac{(e^{(I\,w)} - 1 - I\,w)\,e^{(-I\,w)}}{w^2} - \frac{I\,(e^{(I\,w)} - 1)\,e^{(-2\,I\,w)}}{w} - \frac{(I\,e^{(I\,w)}\,w - e^{(I\,w)} + 1)\,e^{(-3\,I\,w)}}{w^2}$$

```
> ampl:=2*abs(F(w))/2/Pi;
```

$$ampl := \frac{\left| \frac{(e^{(I\,w)} - 1 - I\,w)\,e^{(-I\,w)}}{w^2} + \frac{I\,(e^{(I\,w)} - 1)\,e^{(-2\,I\,w)}}{w} + \frac{(I\,e^{(I\,w)}\,w - e^{(I\,w)} + 1)\,e^{(-3\,I\,w)}}{w^2} \right|}{\pi}$$

```
> ensp:=ampl^2:
> p1:=plot(ampl,w=-20..20,color=blue):
> p2:=plot(ampl*ampl,w=-20..20,color=red):
> display(p1,p2);
```



amplitude $\frac{2|F(\omega)|}{2\pi}$

energy = (amplitude)^2

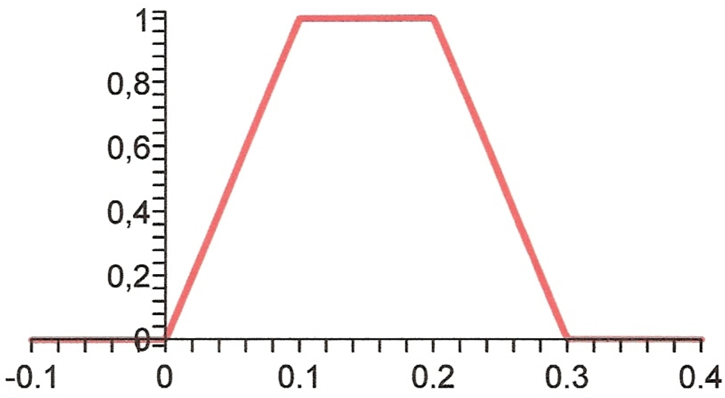
165 b)

```
> restart:with(plots):
Warning, the name changecoords has been redefined
```

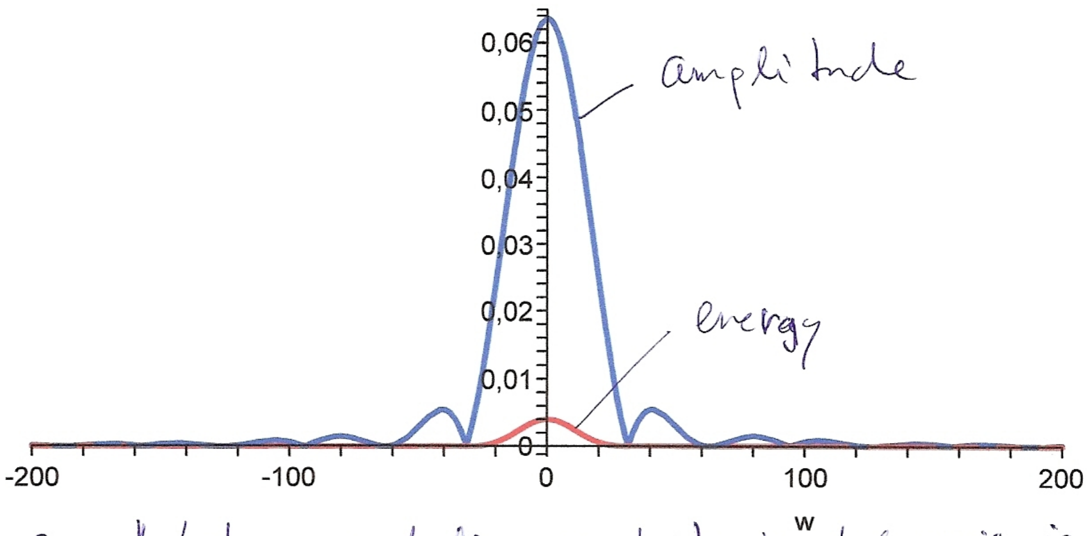
```
> f:=piecewise(x<0,0,x>0 and x<.1, 10*x, x>.1 and x<.2, 1, x>.2 and x<.3
  3-10*x,x>3,0);
```

$$f := \begin{cases} 0 & x < 0 \\ 10\,x & 0 < x \text{ and } x < 0.1 \\ 1 & 0.1 < x \text{ and } x < 0.2 \\ 3 - 10\,x & 0.2 < x \text{ and } x < 0.3 \\ 0 & 3 < x \end{cases}$$

```
> plot(f,x=-0.1..0.4);
```



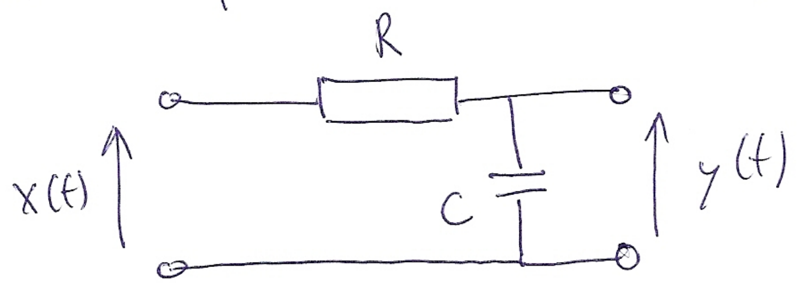

```
> F(w):=int(f*exp(-I*x*w),x=0..0.3):
> ampl:=2*abs(F(w))/2/Pi:
> ensp:=ampl^2:
> p1:=plot(ampl,w=-200..200,color=blue):
> p2:=plot(ampl*ampl,w=-200..200,color=red):
> display(p1,p2);
```



"surface" between red line and horizontal axis is about 10 times smaller with respect to case a)

Application of Fourier transform to the solution of ordinary differential equations

Example: We consider analog RC filter with input $x(t)$ and output $y(t)$



given by ODE $\quad RC\,y'(t) + y(t) = x(t)$

and we are interested in particular solution (i.e. without transient)

$$\mathcal{F} \Big| \quad RC\,y'(t) + y(t) = x(t)$$

$$RC\,i\omega\,Y(\omega) + Y(\omega) = X(\omega)$$

where
$$Y(\omega) = \mathcal{F}\{y(t)\}$$
$$X(\omega) = \mathcal{F}\{x(t)\}$$

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$$Y(\omega) = \underbrace{\frac{1}{1+i\omega RC}}_{=H(\omega)\ \cdots\ \text{transfer function}} X(\omega)$$

$$\rightarrow \quad \underline{y(t) = h(t) * x(t)}, \quad h(t) = \mathcal{F}^{-1}\{H(\omega)\}$$

Let's see the gain of filter $H(f) = \dfrac{1}{1+i\underbrace{2\pi RC}_{=\alpha}\cdot f} =$

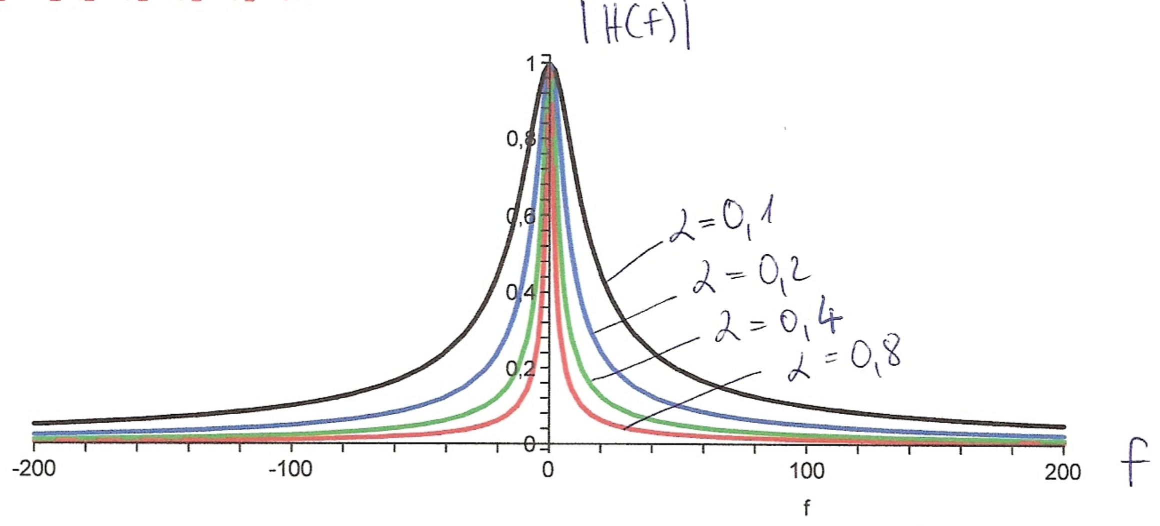
$= \dfrac{1}{1+i\alpha f}$, f is frequency

```
> restart:
> gain(f):=abs(1/(1+I*f*alpha));
```

$$gain(f) := \frac{1}{|1+I\,f\,\alpha|}$$

gain $= |H(f)|$

```
> with(plots):
Warning, the name changecoords has been redefined
> p1:=plot(subs(alpha=0.1,gain(f)),f=-200..200,color=black):
> p2:=plot(subs(alpha=0.2,gain(f)),f=-200..200,color=blue):
> p3:=plot(subs(alpha=0.4,gain(f)),f=-200..200,color=green):
> p4:=plot(subs(alpha=0.8,gain(f)),f=-200..200,color=red):
> display(p1,p2,p3,p4);
```



Application of Fourier transform to the solution of partial differential eq.

Example: Consider the initial value problem for hyperbolic equation of 2nd order.

$$\rho \frac{\partial^2 u(x,t)}{\partial t^2} - \tau \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \qquad (*)$$

for $[x,t] \in (-\infty,\infty) \times (0,\infty)$

and initial conditions

$$\left. \begin{aligned} u(x,0) &= u_0(x) \\ \frac{\partial u}{\partial t}(x,0) &= v_0(x) \end{aligned} \right\} \quad x \in (-\infty,\infty)$$

We apply Fourier transform in variable x to equation $(*)$:

$$\rho \frac{\partial^2 U(\omega,t)}{\partial t^2} + \tau \omega^2 U(\omega,t) = 0$$

where $U(\omega,t) = \mathcal{F}\{u(x,t)\}$

$$\rightarrow \quad U(\omega,t) = C_1(\omega)\cos(\sigma\omega t) + C_2(\omega)\sin(\sigma\omega t)$$

where $\sigma = \sqrt{\frac{\tau}{\rho}}$

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$$U(\omega,0) = \underline{C_1(\omega) = \mathcal{F}\{u_0(x)\}}$$

and

$$\frac{\partial U}{\partial t}(\omega,t) = -C_1(\omega)\sin(\sigma\omega t)\cdot\sigma\cdot\omega + $$
$$+ C_2(\omega)\cos(\sigma\omega t)\cdot\sigma\omega$$

$$\frac{\partial U}{\partial t}(\omega,0) = \underline{C_2(\omega)\cdot\sigma\cdot\omega = \mathcal{F}\{v_0(x)\}}$$

$\vdots$

$$\underline{u(x,t) = \mathcal{F}^{-1}\{U(\omega,t)\}}$$

<u>Remark:</u> If $f(x)$ is even function, it is possible to derive Fourier integral with cosine function, which gives cosine Fourier transform

$$\mathcal{F}_c\{f(x)\} = F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos(\omega x)\,dx$$

$$\mathcal{F}_c^{-1}\{F_c(\omega)\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\omega)\cos(\omega x)\,d\omega$$

and if $f(x)$ is odd we can define

sine Fourier transform

$$\mathcal{F}_s\{f(x)\} = F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin(\omega x)\,dx$$

$$\mathcal{F}_s^{-1}\{F_s(\omega)\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\omega)\sin(\omega x)\,d\omega$$

Discrete Fourier transform

We consider piecewise continuous function $f(t)$ for $t \in \langle 0; a \rangle$. We can define periodic extension to periodic function with perood a. We know, we can write
$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{i 2\pi k \frac{t}{a}} .$$

Let's sample the function $f(t)$ at discrete time $t_j = j \frac{a}{N}$, $j = 0, \dots, N$

$$\Rightarrow \quad f_j = f(t_j) = \sum_{k=\infty}^{\infty} c_k e^{i 2\pi k \frac{1}{\cancel{a}} j \frac{\cancel{a}}{N}}$$

we have $N+1$ of values f_j, therefore we can find $N+1$ of coefficients c_k only, i.e.

$$f_j = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} c_k e^{i 2\pi k j \frac{1}{N}} , \quad j = 0, \dots, N$$

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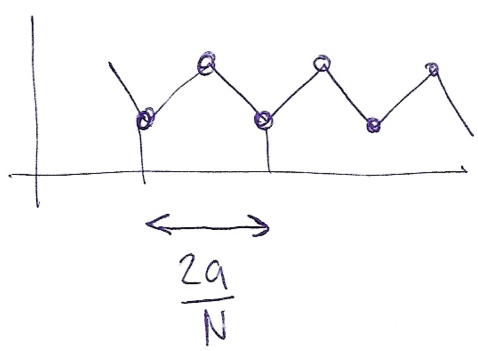
let's denote

$$i \cdot \frac{2\pi k}{a} \cdot j\frac{a}{N} = i \cdot \omega_k \cdot t_j$$

i.e. the k-th frequency is $\nu_k = \frac{k}{a}$.

$\nu_{k+1} - \nu_k = \frac{1}{a} \Rightarrow$ the difference between k-th and (k+1)-th frequency depends on the length a.

The lowest frequency is the 0-th frequency $\nu_0 = 0$, i.e. constant and the highest frequency is $\frac{N}{2}$-th frequency $\nu_{\frac{N}{2}} = \frac{N}{2a}$ i.e.



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so we can write

$$f_j = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} c_k e^{i\omega_k t_j} =$$

$$= \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} c_k \left(\cos(\omega_k t_j) + i \sin(\omega_k t_j) \right) =$$

$$= c_0 + \sum_{k=1}^{\frac{N}{2}} \Big[(c_k + c_{-k}) \cos(\omega_k t_j) + i (c_k - c_{-k}) \cdot$$

$$\cdot \sin(\omega_k t_j) \Big]$$

let's denote $c_k + c_{-k} = a_k \in \mathbb{R}$

$$i(c_k - c_{-k}) = b_k \in \mathbb{R}$$

$$c_k = a_k - i b_k$$

$$c_{-k} = a_k + i b_k$$

$$\Rightarrow c_{-k} = \overline{c_k}$$

As we have already seen at Fourier series, it is possible to write

$$f_j = c_0 + \sum_{k=1}^{N/2} A_k \sin(\omega_k t_j + \varphi_k)$$

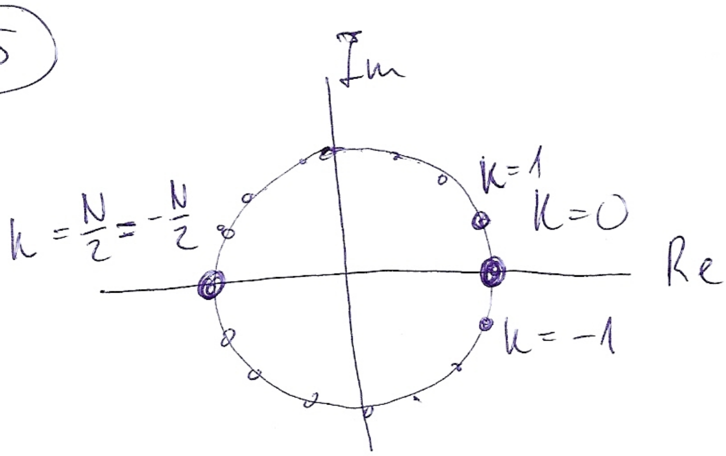
where $A_k = 2|c_k|$, $k = 0, \dots, \frac{N}{2}$ is amplitude and φ_k is phase shift

How to find coefficients c_k for given sequence f_j?

We know that

$$f_j = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} c_k e^{i \frac{2\pi k}{N} j}$$ where

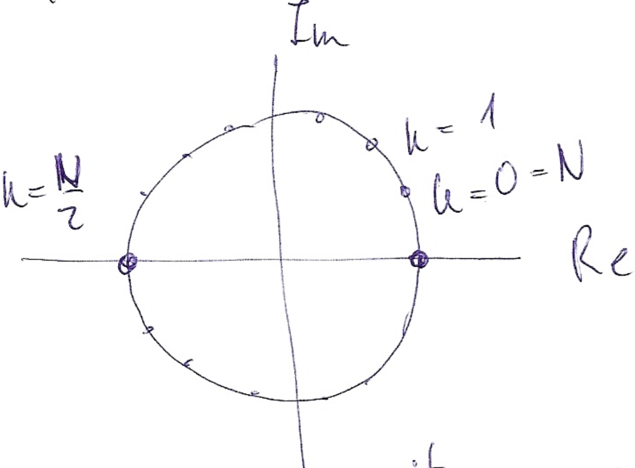
$e^{i\frac{2\pi k}{N}}$, $k = -\frac{N}{2}, \dots, \frac{N}{2}$ represents N points on unit circle in complex plane



If we apply different numbering

$e^{i\frac{2\pi k}{N}}$, $k = 0, \ldots, N$ we get the

same points in complex plane



So we can write

$$f_j = \sum_{k=0}^{N} \tilde{c}_k \, e^{i\frac{2\pi k}{N} j}$$

Since $e^{i\frac{2\pi \cdot 0}{N} j} = e^{i\frac{2\pi N}{N} j}$, we have

only N independent functions $e^{i\frac{2\pi k}{N}}$,

therefore we consider

(176)

$$f_j = \sum_{k=0}^{N-1} \tilde{c}_k \, e^{i \frac{2\pi k}{N} j}, \quad j = 0, \dots, N-1 \qquad (*)$$

and that $f_0 = f_N$, i.e. one period is given by f_j, $j = 0, \dots, N-1$.

Between old coefficients c_k and new $\tilde{c}_k$ is following relation

$$c_{-\frac{N}{2}} = \tilde{c}_{\frac{N}{2}}$$

$$c_{-\frac{N}{2}+1} = \tilde{c}_{\frac{N}{2}+1}$$

$$\vdots$$

$$c_{-1} = \tilde{c}_{N-1}$$

$$c_0 = \tilde{c}_0$$

$$c_1 = \tilde{c}_1$$

$$\vdots$$

$$c_{\frac{N}{2}} = \tilde{c}_{\frac{N}{2}}$$

since $c_k = \overline{c_{-k}} \Rightarrow \tilde{c}_{N-k} = \overline{\tilde{c}_k}$

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Equations (*) can be written in vectorial form

$$\underbrace{\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ \vdots \\ f_{N-1} \end{pmatrix}}_{\mathbb{Z}} = \tilde{c}_0 \underbrace{\begin{pmatrix} w_0 \\ w_0 \\ \vdots \\ \vdots \\ w_0 \end{pmatrix}}_{\mathbb{W}_0} + \cdots + \tilde{c}_K \underbrace{\begin{pmatrix} w_0 \\ w_{K\cdot 1} \\ \vdots \\ \vdots \\ w_{K(N-1)} \end{pmatrix}}_{\mathbb{W}_K} + \cdots + \tilde{c}_{N-1} \underbrace{\begin{pmatrix} w_0 \\ w_{(N-1)\cdot 1} \\ \vdots \\ \vdots \\ w_{(N-1)\cdot(N-1)} \end{pmatrix}}_{\mathbb{W}_{N-1}}$$

where $w_{K\cdot j} = e^{i \frac{2\pi}{N} K\cdot j}$

i.e. the vector $\mathbb{Z}$ is expressed as the linear combination of vectors

$\mathbb{W}_0, \mathbb{W}_1, \ldots, \mathbb{W}_{N-1}$.

<u>Theorem:</u> $\mathbb{W}_K \cdot \mathbb{W}_j = \begin{cases} N & , j = k \\ 0 & , j \neq k \end{cases}$

scalar product

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<u>Proofs</u>

$$W_k \cdot W_j = \sum_{l=0}^{N-1} \underbrace{w_{k\cdot l} \cdot \overline{w_{j\cdot l}}}_{\text{note the definition of scalar product for complex vectors}} =$$

$$= \sum_{l=0}^{N-1} w_{kl}\, w_{-j\cdot l} = \sum_{l=0}^{N-1} w_{(k-j)\cdot l} =$$

$$= \sum_{l=0}^{N-1} (w_{k-j})^{l} = \frac{(w_{k-j})^{N}-1}{w_{k-j}-1} =$$

$$= \frac{w_{(k-j)\cdot N} - 1}{w_{k-j} - 1} = 0 \quad \text{for } k \neq j.$$

(since $w_{(k-j)\cdot N} = 1$ and $w_{k-j} \neq 1$ for $k \neq j$)

let's see particular case $k=j$

$$W_k \cdot W_k = \sum_{l=0}^{N-1} w_{(k-k)\cdot l} = \sum_{l=0}^{N-1} w_0 = \sum_{l=0}^{N-1} 1 = N$$

$\square$

<u>Conclusion:</u> Vectors $\mathbb{W}_0, \mathbb{W}_1, \dots, \mathbb{W}_{N-1}$ constitutes orthogonal basis of N-dimensional vectorial space and therefore any vector $\mathbb{Z}$ (which represents our sampled function) can be written as linear combination of $\mathbb{W}_0, \mathbb{W}_1, \dots, \mathbb{W}_{N-1}$:

$$\mathbb{Z} = c_0 \mathbb{W}_0 + c_1 \mathbb{W}_1 + \dots + c_{N-1} \mathbb{W}_{N-1},$$

which can be written like

$$\mathbb{Z} = \mathbb{F}_N \cdot \mathbb{C}, \quad \text{where}$$

$$\mathbb{F}_N = \begin{pmatrix} | & | & & | \\ \mathbb{W}_0 & \mathbb{W}_1 & \cdots & \mathbb{W}_{N-1} \\ | & | & & | \end{pmatrix} \quad \text{is matrix } (N \times N)$$

and $$\mathbb{C} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{pmatrix} \quad \text{is vector } (N \times 1)$$

$$\Rightarrow \quad \mathbb{C} = \mathbb{F}_N^{-1} \cdot \mathbb{Z}$$

<u>Definition:</u> The vector $\mathbb{C} = \mathbb{F}_N^{-1} \cdot \mathbb{Z}$ is the <u>discrete Fourier transform</u> of original vector $\mathbb{Z}$. We write $\mathbb{C} = \mathcal{F}_D\{\mathbb{Z}\}$.

<u>Theorem:</u> $\mathbb{F}_N^{-1} = \frac{1}{N}\overline{\mathbb{F}_N}$.

<u>Proof:</u>

$$\mathbb{F}_N \cdot \overline{\mathbb{F}_N} = \begin{pmatrix} | & | & & | \\ W_0 & W_1 & \cdots & W_{N-1} \\ | & | & & | \end{pmatrix} \cdot \begin{pmatrix} | & | & & | \\ \overline{W_0} & \overline{W_1} & \cdots & \overline{W_{N-1}} \\ | & | & & | \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} \text{---} & W_0 & \text{---} \\ \text{---} & W_1 & \text{---} \\ & \vdots & \\ \text{---} & W_{N-1} & \text{---} \end{pmatrix}}_{\text{since } \mathbb{F}_N = \mathbb{F}_N^T} \cdot \begin{pmatrix} | & | & & | \\ \overline{W_0} & \overline{W_1} & \cdots & \overline{W_{N-1}} \\ | & | & & | \end{pmatrix} =$$

$$= \begin{pmatrix} N & 0 & 0 & \cdots & 0 \\ 0 & N & 0 & \cdots & 0 \\ \vdots & & \ddots & & 0 \\ & & & 0 & N \end{pmatrix} = N \cdot I \Rightarrow \mathbb{F}_N^{-1} = \frac{1}{N}\overline{\mathbb{F}_N}$$

$\square$