(1)

Complex numbers

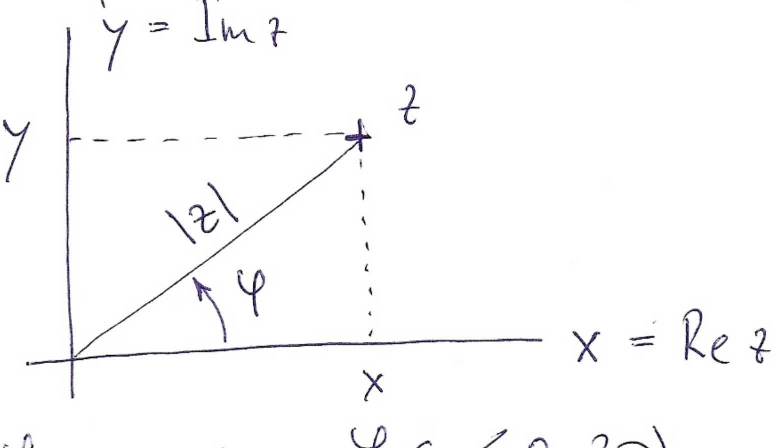
- complex numbers were invented in the Middle Age - formulas for cubic equation derived from solution back to the equation

- complex number $z = x + iy$, where x and y are real numbers and they are called: real part of z $\quad Re(z) = x$

  imaginary part of z $\quad Im(z) = y$

  and i is complex unit, $i = \sqrt{-1}$.

- complex plane



where $\varphi = \arg z$, $\varphi \in \langle 0; 2\pi)$

$|z| = \sqrt{(Re\, z)^2 + (Im\, z)^2} = \sqrt{x^2+y^2}$ ... absolute value of z or modulus of z

There is also $\text{Arg}\, z = \arg z + 2k\pi$,

$k = \ldots, -2, -1, 0, 1, 2, \ldots$

(which is not unique).

- from complex plane representation follows trigonometric (polar) representation

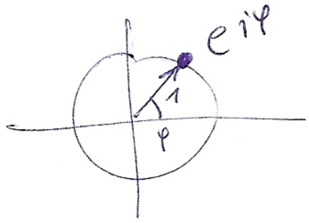
$$z = |z| \cdot (\cos\varphi + i \cdot \sin\varphi)$$

- exponential representation

$$z = |z| \cdot e^{i\varphi} = |z| (\cos\varphi + i \sin\varphi)$$

where $|e^{i\varphi}| = 1$

$\arg(e^{i\varphi}) = \varphi$



- complex arithmetic

addition and subtraction e.g.

$$(1+2i) + (3-3i) = 4-i$$
$$(1+2i) - (3-3i) = -2+5i$$

rem: can be also explained as addition or ~~subont~~ subtraction of

③

"vectors" in complex plane

- multiplication also intuitive

$(1+2i)\cdot(3-3i) = 3-3i+6i-6(i)^2 =$

$= 9+3i$

in complex plane $z_1 \cdot z_2 = z$

$|z| = |z_1| \cdot |z_2|$

$\arg(z) = \arg(z_1) + \arg(z_2)$

- division

$$\frac{1+2i}{3-3i} \cdot \frac{3+3i}{3+3i} = \frac{(1+2i)\cdot(3+3i)}{9+9} = \ldots$$

complex conjugate
(číslo komplexně sdružené)

Example: complex arithmetic with Maple

```
evalc((2+3*I)/(3-4*I));
```

or

```
(2+3*I)/(3-4*I);
simplify(%);
```

Complex functions

complex / real > function of < complex / real > variable

Example: e^{ix} $\quad R \to C$

```
f:= exp(I*x);
Re(f);
Im(f);
```

Ex.2 e^{1+2i} $\quad C \to C$

```
evalc(exp(1+2*I));
evalf(%);
```

--- in decimal numbers

⑤

complex functions of complex variable

- e^z, $\cosh z$, $\sinh z$, $\cos z$, $\sin z$, where z is complex are defined as the sum of series, e.g.

$$e^z = \underbrace{1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots}_{\text{the same like for real funtcion}} = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

- we can define properties like
  even (sudá) function $f(-z) = f(z)$
  odd (lichá) function $f(-z) = -f(z)$

- some functions are not unique (single valued) functions

  e.g. $\text{Arg}\, z$

Ex.: complex logarithm

suppose $z = |z| \cdot e^{i(\varphi + 2\pi k)}$,

z is complex, k is integer, φ is real

6

$$\ln(z) = \ln\left(|z| \cdot e^{i(\varphi + 2k\pi)}\right)$$

$$\ln z = \ln|z| + i(\varphi + 2k\pi)$$

we define also single valued function

$$\mathrm{Ln}\, z = \ln|z| + i\varphi$$

Ex: n-th root of z

$$\sqrt[n]{z} = \sqrt[n]{|z|}\left(\cos\frac{\varphi + 2\pi k}{n} + i\sin\frac{\varphi + 2k\pi}{n}\right)$$

$k = 0, 1, 2, \ldots, n-1$

(proove by $\left(\sqrt[n]{z}\right)^n = \ldots = z$ □)

7

Complex differentiation

The derivative of function $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z},$$

where z and Δz are complex.

Suppose $z = x + iy$, $\Delta z = \Delta x + i\Delta y$, where $x = \mathrm{Re}\, z$, $y = \mathrm{Im}\, z$

and $f(z) = u(x,y) + i\,v(x,y)$, where

u and v are real functions

1) let's compute $f'(z)$ for $\Delta y = 0$

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + i\,v(x+\Delta x, y) - u(x,y) - i\,v(x,y)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} =$$

$$= \underline{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}$$

2) if we choose another direction, eg. $\Delta x = 0$ we get

$$f'(z) = \lim_{i\Delta y \to 0} \frac{u(x,y+\Delta y) + i v(x,y+\Delta y) - u(x,y) - i v(x,y)}{i \Delta y}$$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

We want $f'(z)$ independent of direction

i.e.

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}}$$

Cauchy - Riemann equations

(C-R. podmínky)

→ necessary condition for diferentiability

Theorem: If Cauchy - Riemann equations are satisfied and $u(x,y)$ and $v(x,y)$ have continuous derivatives, then function $f(z)$ is diferentiable (also holomorphic or analytic).

Remark: If $f(z)$ is holomorphic, then

(9)

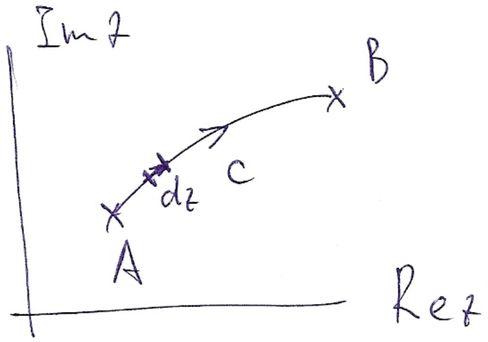
derivative of any order of $f(z)$ exists.

Exercise: Show that $f(z) = e^z$ is analytic function.

```
assume (x, real, y, real):
z:= x + I*y;          -- definition of complex number

u:= Re(exp(z));
v:= Im(exp(z));

vy:= diff(v, y);
ux:= diff(u, x);

uy:=
vx:=
```

(10)

Line integral



curve c described as

$x = \varphi(t)$

$y = \psi(t)$

$t \in \langle \alpha, \beta \rangle$

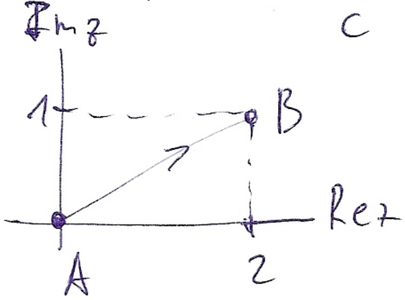
then points on c are $z(t) = \varphi(t) + i\psi(t)$

and $dz = \dot{z}(t)\,dt = \left[\dot{\varphi}(t) + i\,\dot{\psi}(t)\right] dt$

then $\displaystyle\int_c f(z)\,dz = \int_\alpha^\beta f(z(t))\left(\dot{\varphi}(t) + i\,\dot{\psi}(t)\right) dt$

Ex: compute $\displaystyle\int_c z \cdot \mathrm{Re}\, z\, dz$ along curve

c



rectification of c:

$x = t$

$y = t/2$

$t \in \langle 0; 2 \rangle$

$\Rightarrow z(t) = t + i\,\frac{t}{2}$

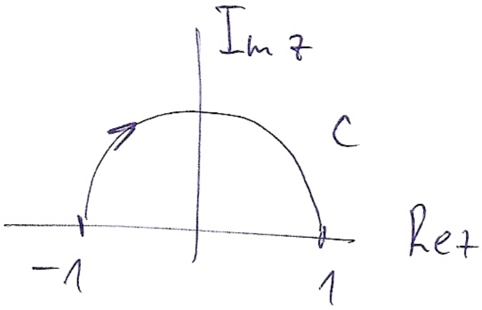
(11)

```
restart:
assume (t, real):
z:= t+ I*t/2;
dz:= diff (z,t):
int ( z*Re(z) * dz , t = 0..2);
```

colon

semicolon

Ex.: $\int_C |z| \, dz$



$x = \cos t$

$y = \sin t$

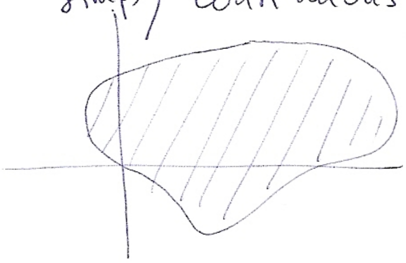
$t \in \langle 0 ; \pi \rangle$ negative orientation

```
restart:
assume (t, real):
z:= cos (t) + I* sin (t);
dz:= diff (z,t);
int (abs (z) * dz , t = Pi..0);
```

Cauchy theorem

Suppose we have function $f(z)$, which is analytic in some simply continuous domain G in complex plane.

simply continuous domain



not simply continuous domain

(Domain G is simply continuous if for any closed curve c hold that $\text{int}\, C \subset G$)

and closed curve $c \subset G$, then

$$\boxed{\oint_C f(z)\,dz = 0}$$

Proof:

$$\oint_C f(z)\,dz = \oint_C \big(u(x,y) + i\,v(x,y)\big)\cdot\big(dx + i\,dy\big) =$$

$$= \oint_C (u\,dx - v\,dy) + i \oint_C (v\,dx + u\,dy) =$$

$$= \oint_C (v,u)(-dy, dx) + i \oint_C (-u,v)(-dy, dx) =$$

(13)

$$= \iint_{\text{int}\, c} \underbrace{\text{div}\,(v,u)}_{=0}\, dS + i \iint_{\text{int}\, c} \underbrace{\text{div}\,(-u,v)}_{=0}\, dS = 0 \qquad \square$$

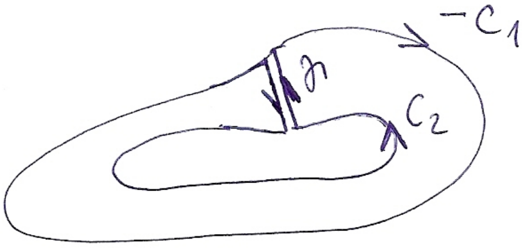
Green's theorem: $\iint_{\text{int}\, C} \text{div}\,(u,v)\, dS = \oint_C (u,v)\, \vec{n}_0\, ds =$

$$= \oint_C (-u\,dy + v\,dx)$$

Corollary:

Suppose we have two closed curves c_1 and c_2

and that function $f(z)$ is analytic on domain G, where $G = \text{int}\, c_1 \cap \text{ext}\, c_2$, then



$$\int_{-c_1} f(z)\,dz + \int_{\gamma} f(z)\,dz + \int_{c_2} f(z)\,dz -$$

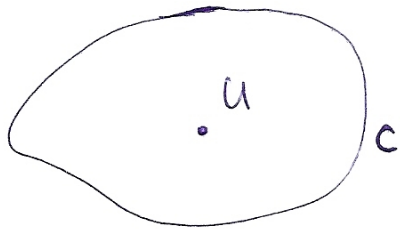
$$+ \int_{-\gamma} f(z)\,dz = 0$$

$$\Rightarrow \boxed{\oint_{c_1} f(z)\,dz = \oint_{c_2} f(z)\,dz}$$

(14)

Cauchy integral formula

Suppose we have closed curve C and function $f(z)$ analytic on c and inside int C.



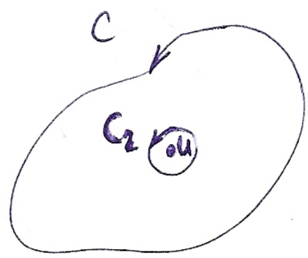
We define function

$$g(z) = \frac{f(z)-f(u)}{z-u},$$

which is not analytic in point $z = u$.

From previous corollary

$$\oint_C g(z)\,dz = \oint_{C_2} g(z)\,dz$$



Function $g(z)$ is not analytic in $z = u$, but we can define $g(u) = f'(u) \Rightarrow g(u)$ is then continuous inside int C. If we shrink C_2 in order to (area of int C_2) $\to 0$, then

$$\oint_{C_2} g(z)\,dz = 0 \Rightarrow \oint_C g(z)\,dz = 0$$

i.e.

$$\oint_C \frac{f(z)-f(u)}{z-u}\,dz = 0$$

15

then $\oint_C \frac{f(z)}{z-u}\,dz = f(u) \oint_C \frac{dz}{z-u}$

thanks to Cauchy theorem
we can consider
circle with origin $[\alpha, \beta]$
and radius r instead of C

circle: $x = r\cos\varphi + \alpha$
$y = ir\sin\varphi + i\beta$
$\varphi \in \langle 0, 2\pi \rangle$

i.e. $z = re^{i\varphi} + u$
$z' = ire^{i\varphi}$

i.e. $\oint_C \frac{dz}{z-u} = \int_0^{2\pi} \frac{ire^{i\varphi}}{re^{i\varphi}} = i\int_0^{2\pi} d\varphi = 2\pi i$

$$\Rightarrow \boxed{f(u) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-u}\,dz}$$

derive with respect to u inside integral of $f(z)$

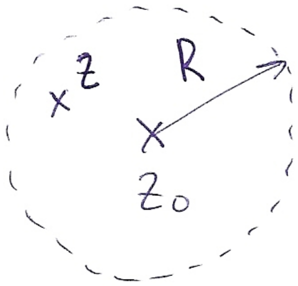
Cauchy integral formula

generally for n-th derivative in point $z = u$

$$\boxed{f^{(n)}(u) = \frac{n!}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-u)^{n+1}}}$$

Taylor series

Suppose $f(z)$ analytic for $|z-z_0|<R$,
R is possitive real number.



From Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(u)}{u-z}\, du$$

C is closed curve inside circle

where

$$\frac{1}{u-z} = \frac{1}{u-z_0-(z-z_0)} = \underbrace{\frac{1}{u-z_0}}_{=a} \frac{1}{1-\underbrace{\frac{z-z_0}{u-z_0}}_{=q}}$$

(we know $a\frac{1}{1-q} = \sum_{k=0}^{\infty} a q^k$ for $|q|<1$)

$$|q| = \left|\frac{z-z_0}{u-z_0}\right| < 1 \rightarrow |z-z_0| < |u-z_0|$$