Numerical solution of inviscid and viscous flows using modern schemes and quadrilateral or triangular mesh

J. Fäurst and K. Kozel

Dedicated to Prof. J. Nečas on the occasion of his 70th birthday.

Abstract

This contribution deals with the modern finite volume schemes solving the Euler and Navier-Stokes equations for transonic flow problems. We will mention the TVD theory for first order schemes and some numerical examples obtained by 2D central and upwind schemes for 2D transonic flows in the GAMM channel or through the SE 1050 turbine of Škoda Plzeň. The TVD MacCormack method is extended to a 3D method for solving flows through turbine cascades. Numerical examples of unsteady transonic viscous (laminar) flows through the DCA 8% cascade are also presented for $Re = 4600$.

Next, a new finite volume implicit scheme is presented for the case of unstructured meshes (with both triangular and quadrilateral cells) and inviscid compressible flows through the GAMM channel as well as the SE 1050 turbine cascade.

Key words: transonic flow, Euler equations, Navier-Stokes equations, numerical solution, TVD, ENO

MSC classification: 65C20, 65N30, 76H05

1 Mathematical model

We consider the system of 2D Navier-Stokes equations for compressible medium in conservative form:

$$\begin{align*}
W_t + F_x + G_y &= R_x + S_y, \\
W &= \text{col} ||\rho, \rho u, \rho v, e||, \\
F &= \text{col} ||\rho u, \rho u^2 + p, \rho u v, (e + p)u||, \\
G &= \text{col} ||\rho v, \rho u v, \rho v^2 + p, (e + p)v||, \\
R &= \text{col} ||0, \tau_{11}, \tau_{12}, u \tau_{11} + v \tau_{12} + k T_x||, \\
S &= \text{col} ||0, \tau_{21}, \tau_{22}, u \tau_{21} + v \tau_{22} + k T_y||,
\end{align*}$$

where $\rho$ is the density, $(u,v)$ the velocity vector, $e$ the total energy per unit volume, $\mu$ the viscosity coefficient, $k$ is the heat conductivity, $p$ is the pressure, $\gamma$ is the adiabatic coefficient, and the components of the stress tensor $\tau$ are

$$\tau_{11} = \mu \left(\frac{4}{3} u_x - \frac{2}{3} v_y\right), \quad \tau_{21} = \tau_{12} = \mu (u_y + v_x), \quad \tau_{22} = \mu \left(-\frac{2}{3} u_x + \frac{4}{3} v_y\right).$$

The 2D Euler equations are obtained from the Navier-Stokes equations by setting $\mu = k = 0$. The system of 3D Euler equations is written also in conservative form (here $w$ is the third component of the velocity vector):

$$\begin{align*}
W_t + F_x + G_y + H_z &= 0, \\
p &= (\gamma - 1) \left(e - \frac{1}{2} \rho (u^2 + v^2)\right), \\
F &= \text{col} ||\rho u, \rho u^2 + p, \rho u v, (e + p)u||, \\
G &= \text{col} ||\rho v, \rho u v, \rho v^2 + p, \rho v w, (e + p)v||, \\
H &= \text{col} ||\rho w, \rho u w, \rho v w, \rho w^2 + p, (e + p)w||.
\end{align*}$$

1.1 Boundary conditions

We assume four types of boundary conditions:

Inlet: At the inlet we prescribe the direction of the velocity (by the inlet angle for the 2D case and by 2 angles for the 3D case), the value of the stagnation density $\rho_0$ and the stagnation pressure $p_0$. We extrapolate the static pressure $p$ from inside and compute the other required quantities using the following relations between the stagnation and the static quantities:

$$p_0 = p \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}, \quad \rho_0 = \rho \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma - 1}}.$$
where $M$ is the local Mach number defined by $M = \sqrt{u^2 + v^2 + w^2}/\sqrt{\gamma \rho}$. For the Navier-Stokes equations we assume $\partial T/\partial n = 0$ and $W_\infty$ given.

**Outlet:** At the outlet we prescribe the value of the static pressure $p$ and extrapolate the values of the density $\rho$ and of the velocity vector from the flow field. For the viscous flows we assume again $\partial T/\partial n = 0$.

**Solid wall:** Here we prescribe the non-permeability condition $(\vec{u} \cdot \vec{n} = 0)$ for the inviscid case or $\vec{u} = 0$ for the case of viscous flows. Here we assume also the adiabatic walls (i.e. $\partial T/\partial n = 0$ where $T$ is the temperature).

**Periodicity:** Here we prescribe the periodical condition for all components of the vector of unknowns $W$.

## 2 Numerical methods

### 2.1 TVD schemes for one-dimensional scalar case

The theory for full nonlinear systems is very complicated and we restrict the analysis of the numerical method only to the scalar case. We assume the initial value problem for the one-dimensional scalar equation

$$u_t + f(u)_x = 0$$

with the initial condition $u(x, 0) = u_0(x)$. This initial value problem is solved for $(x, t) \in \mathbb{R} \times \mathbb{R}_+$. However the theoretical results are not straightforward applicable to nonlinear systems and to bounded domains, the simple model case is still important for understanding some properties of the numerical methods and as a hint for constructing good numerical methods for systems.

We approximate the weak solution to (5) by a piecewise constant function $U(x, t) = u^0_i$ for $(i - 1)\Delta x < x \leq i\Delta x$ and $(n - 1)\Delta t < t \leq n\Delta t$ where $\Delta x$ and $\Delta t$ are the mesh spacings in the space and time variables. The initial condition $u^0_i$ is computed as

$$u^0_i = \frac{1}{\Delta x} \int_{(i-1)\Delta x}^{i\Delta x} u_0(x) \, dx. \quad (6)$$

The values $u^0_i$ are computed for $n > 0$ using the following explicit numerical scheme in so the called conservation form:

$$u^{n+1}_i = u^n_i - \frac{\Delta t}{\Delta x} \left[ \tilde{f}(u^n_{i-p}, u^n_{i-p+1}, \ldots, u^n_{i+q}) - \tilde{f}(u^n_{i-p-1}, u^n_{i-p}, \ldots, u^n_{i+q-1}) \right] \quad (7)$$

where $\tilde{f}$ is a continuous function of $p + q + 1$ parameters called the numerical flux function which approximates the physical flux function $f$ in the following sense:

$$\forall v : \tilde{f}(v, v, \ldots, v) = f(v). \quad (8)$$

The analysis of the above mentioned method is complicated even in the one-dimensional scalar case because of the nonlinearity of the flux function $f$ (and consequently $\tilde{f}$). Nevertheless, there exist good theoretical backgrounds for certain subclasses of the general method. Namely, the following facts have been proved:

- the convergence towards the unique (so called viscosity vanishing\(^1\)) weak solution of (5) for the class of monotone methods,

- the convergence towards the unique weak solution (5) for the class of weakly TV bounded methods (see [2] for details),

- the convergence towards the set of all weak solutions\(^2\) of (5) for the class of TVD methods (see [11], [10]).

However the theory for the monotone methods is very strong and is easily extensible to the multidimensional case, the monotone methods are at most of the first order of accuracy. Therefore we prefer the class of TVD methods which is defined as follows:

\(^1\)The viscosity vanishing solution is obtained as a limit of solutions to the problems given by the equation $u_\varepsilon + f(u'_\varepsilon)_x = \varepsilon u_{\varepsilon,xx}$ for $\varepsilon \rightarrow 0+$.

\(^2\)The uniqueness of the solution follows from Coquel's-LeFloch's theorem for the weakly TV bounded methods.
Definition 1 The numerical method (7) is called total variation diminishing (or simply TVD), if and only if for each numerical approximation \( U \)

\[
TV(u^{n+1}) \leq TV(u^n) = \sum_{i=-\infty}^{+\infty} |u_{i+1}^n - u_i^n|.
\] (9)

Let us consider a general one-dimensional method of the form

\[
u_{i+1}^{n+1} = \frac{\Delta t}{\Delta x} \left( f(u_i^n) - f(u_{i-1}^n) \right) + D_{i+\frac{1}{2}}(u_{i+1}^n - u_i^n),
\] (10)

then the following theorem of Harten \[11]\ can be used to check the TVD property.

Lemma 1 (Harten, 1983) Let the following conditions be fulfilled \( \forall i \in \mathbb{Z} \):

\[
C_{i-\frac{1}{2}} \geq 0, \quad D_{i+\frac{1}{2}} \geq 0, \quad C_{i+\frac{1}{2}} + D_{i+\frac{1}{2}} \leq 1.
\] (11)

Then the numerical method (10) is TVD.

2.2 Extension to the 2D scalar case

The scalar initial value problem for the 2D problem is given by the equation \( u_x + f(u)_x + g(v)_y = 0 \) with the initial condition \( u(x, y, 0) = u_0(x, y) \). We again solve this problem in the unbounded domain \( (x, y, t) \in \mathbb{R} \times \mathbb{R} \times R_+ \).

For a two-dimensional scalar case we analyzed in \[6\] a class of monotone schemes (especially the so-called \( l^1 \)-contractive schemes) and published in \[8\] the following lemma:

Lemma 2 Let there exist functions \( \tilde{f}_k \) of \( 2(p + q + 1) \) parameters such that

\[
\tilde{f}(u_{i-p}^n, ..., u_{i+q}^n) - \tilde{f}(v_{i-p}^n, ..., v_{i+q}^n) = \sum_{k=-p}^{q} \tilde{f}_k(u_{i-p}^n, ..., u_{i+q}^n, v_{i-p}^n, ..., v_{i+q}^n)(u_{i+k}^n - v_{i+k}^n)
\] (17)

for arbitrary \( u \) and \( v \). If the functions \( \tilde{f}_k \) satisfy the conditions

\[
\tilde{f}_0(u_{i-p}^n, ..., u_{i+q}^n, v_{i-p}^n, ..., v_{i+q}^n) - \tilde{f}_1(u_{i-p-1}^n, ..., u_{i+q-1}^n, v_{i-p}^n, ..., v_{i+q-1}^n) \leq \frac{\Delta x}{\Delta t}
\] (18)

and for each \( k \neq 0 \) (we set \( \tilde{f}_k = 0 \) for \( k < -p \) and \( k > q \) the relations

\[
\tilde{f}_k(u_{i-p}^n, ..., u_{i+q}^n, v_{i-p}^n, ..., v_{i+q}^n) \leq \tilde{f}_{k+1}(u_{i-p-1}^n, ..., u_{i+q-1}^n, v_{i-p}^n, ..., v_{i+q-1}^n)
\] (19)

are valid, then the scheme (7) is monotone (and hence TVD).
This lemma (as well as its multidimensional variant) is proved for example in [6].

A similar lemma is valid also for multidimensional problems. Let us consider only the two-dimensional case. Let the mesh steps be the same for both coordinates i.e. $\Delta x = \Delta y$. The weak solution is approximated by a piecewise constant function $U(x,y,t) = u_{i,j}^n$ for $(i-1)\Delta x < x \leq i\Delta x$, $(j-1)\Delta y < y \leq j\Delta y$, and $(n-1)\Delta t < t \leq n\Delta t$. The functions $\tilde{f}_k$ and $\tilde{g}_k$ depend in a similar way as in the one-dimensional problem on $2(p+q+1)$ parameters $u_{i-p,j}, u_{i+p,j}, u_{i,j-q}, u_{i,j+q}$ for $\tilde{f}_k$ and $u_{i,j-p}, \ldots, u_{i,j-p+q}, u_{i,j+q} \ldots, u_{i,j+q}$ for $\tilde{g}_k$.

**Lemma 3** Let there exist functions $\tilde{f}_k$ and $\tilde{g}_k$ of $2(p+q+1)$ parameters such that

$$
\tilde{f}(u_{i-p,j}, \ldots, u_{i+q,j}) - \tilde{f}(u_{i-p,j}, \ldots, u_{i+q,j}) = \sum_{k=-p}^{q} \tilde{f}_k(u_{i-p,j}, \ldots, u_{i+q,j})(u_{i+1,j+k} - u_{i+1,j+k}),
$$

$$
\tilde{g}(u_{i,j-p}, \ldots, u_{i,j+q}) - \tilde{g}(u_{i,j-p}, \ldots, u_{i,j+q}) = \sum_{k=-p}^{q} \tilde{g}_k(u_{i,j-p}, \ldots, u_{i,j+q})(u_{i+1,j+k} - u_{i+1,j+k})
$$

for each arbitrary $u$ and $v$. If the functions $\tilde{f}_k$ and $\tilde{g}_k$ satisfy the conditions

$$
\tilde{f}_0(u_{i-p,j}, \ldots, u_{i+q,j}) = \tilde{f}_1(u_{i-p-1,j}, \ldots, u_{i+q-1,j}) + \Delta x \frac{\Delta}{\Delta t}
$$

and for each $k \neq 0$ (we set $\tilde{f}_k = 0$ and $\tilde{g}_k = 0$ for $k < -p$ and $k > q$)

$$
\tilde{f}_k(u_{i-p,j}, \ldots, u_{i+q,j}) \leq \tilde{f}_{k+1}(u_{i-p-1,j}, \ldots, u_{i+q-1,j}), \quad \tilde{g}_k(u_{i,j-p}, \ldots, u_{i,j+q}) \leq \tilde{g}_{k+1}(u_{i,j-p-1}, \ldots, u_{i,j+q-1})
$$

then the scheme

$$
u_{i,j}^{n+1} = u_{i,j}^n - \Delta t \left( \tilde{f}_{i+\frac{1}{2},j} - \tilde{f}_{i-\frac{1}{2},j} \right), \quad \tilde{g}_{i+\frac{1}{2},j}^{n+1} = \tilde{g}(u_{i,j-p}, u_{i,j-p+1}, \ldots, u_{i,j+q})
$$

is monotone (and hence TVD).

### 2.3 TVD schemes for hyperbolic systems

Let us consider a linear system

$$W_t + AW_x = 0.
$$

The solution is now a vector-valued function $W : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^m$ ($m$ is the number of equations) and $A$ is an $m \times m$ constant matrix. The system (25) is called hyperbolic if the matrix $A$ has $m$ real eigenvalues and $m$ linearly independent right eigenvectors. In that case, we can decompose the matrix $A$ into

$$A = R \Lambda R^{-1}
$$

where $R$ is the regular matrix composed of the eigenvectors of $A$ and $\Lambda = \text{diag}(a^{(1)}, \ldots, a^{(m)})$ is the diagonal matrix containing the eigenvalues of $A$.

Let us define a new set of variables by $V = R^{-1}W$ (we call it the characteristic variables). Multiplying the original system (25) by $R^{-1}$ one gets

$$R^{-1}W_t + R^{-1}AW_x = 0
$$

and hence, using the characteristic variables,

$$V_t + \Lambda V_x = 0
$$

which is a set of $m$ independent scalar problems. For each component we can use the TVD MacCormack scheme defined in the previous section. In order to get the TVD MacCormack scheme for the original variables
we multiply the scheme written for the characteristic variables \( V \) by the matrix \( R \). Finally, we get

\[
W_i^{n+\frac{1}{2}} = W_i^n - \frac{\Delta t}{\Delta x} (AW_i^n - AW_{i-1}^n),
\]

\[
\overline{W}_i^{n+\frac{1}{2}} = \frac{1}{2} \left[ W_i^n + W_i^{n+\frac{1}{2}} - \frac{\Delta t}{\Delta x} \left( AW_i^{n+\frac{1}{2}} - AW_{i-1}^{n+\frac{1}{2}} \right) \right],
\]

\[
W_i^{n+1} = \overline{W}_i^{n+\frac{1}{2}} + R \left[ \tilde{G}^+ (\tilde{r}_i^+) + \tilde{G}^- (\tilde{r}_{i+1}^-) \right] R^{-1} (W_i^n - W_{i-1}^n) - \] \[
- R \left[ \tilde{G}^+ (\tilde{r}_{i-1}^+) + \tilde{G}^- (\tilde{r}_i^-) \right] R^{-1} (W_i^n - W_{i-1}^n).
\]

Here \( \tilde{r}_i^{\pm} \) are vectors with \( m \) components

\[
(\tilde{r}_i^+) = \left( (R^{-1}(W_i^n - W_{i-1}^n))^{(i)} / (R^{-1}(W_{i+1}^n - W_i^n))^{(i)} \right),
\]

\[
(\tilde{r}_i^-) = \left( (R^{-1}(W_{i+1}^n - W_i^n))^{(i)} / (R^{-1}(W_i^n - W_{i-1}^n))^{(i)} \right),
\]

where \( r^{(i)} \) denotes the \( i \)-th component of the vector \( r \).

The viscosity coefficients \( G \) are \( m \times m \) diagonal matrices with the elements given by

\[
\tilde{G}^\pm (\tilde{r}_i^\pm)^{(i,j)} = \frac{1}{2} \left[ \phi^{(i)} \Delta t / \Delta x \right] (1 - \phi^{(j)}) [1 - \Phi(\tilde{r}_i^\pm)^{(i,j)}].
\]

In order to avoid the evaluation of eigenvectors of the Jacobian matrix \( A \) we use the so called simplified TVD scheme proposed by D. M. Causon in [1] which is for the case of the one-dimensional nonlinear system written as:

\[
W_i^{n+\frac{1}{2}} = W_i^n - \frac{\Delta t}{\Delta x} (F(W_i^n) - F(W_{i-1}^n)),
\]

\[
\overline{W}_i^{n+\frac{1}{2}} = \frac{1}{2} \left[ W_i^n + W_i^{n+\frac{1}{2}} - \frac{\Delta t}{\Delta x} \left( F(W_i^{n+\frac{1}{2}}) - F(W_{i+1}^{n+\frac{1}{2}}) \right) \right],
\]

\[
W_i^{n+1} = \overline{W}_i^{n+\frac{1}{2}} + \left[ \tilde{G}^+ (\tilde{r}_i^+) + \tilde{G}^- (\tilde{r}_{i+1}^-) \right] (W_{i+1}^n - W_i^n) - \] \[
- \left[ \tilde{G}^+ (\tilde{r}_{i-1}^+) + \tilde{G}^- (\tilde{r}_i^-) \right] (W_i^n - W_{i-1}^n)
\]

with \( \tilde{G}^\pm \) and \( \tilde{r}^\pm \) given by the formulas

\[
\tilde{r}_i^+ = \frac{<W_i^n - W_{i+1}^n, W_{i+1}^n - W_i^n>}{<W_i^n - W_{i+1}^n, W_{i+1}^n - W_i^n>}, \quad \tilde{r}_i^- = \frac{<W_i^n - W_{i-1}^n, W_{i-1}^n - W_i^n>}{<W_i^n - W_{i-1}^n, W_{i-1}^n - W_i^n>},
\]

\[
\tilde{G}^\pm (\tilde{r}_i^\pm) = \left\{ \begin{array}{ll}
\frac{1}{2} C(\nu_i) \left[ 1 - \Phi(\tilde{r}_i^\pm) \right], & \text{for } \nu_i \leq 0.5 \\
\nu_i (1 - \nu_i) & \text{for } \nu_i > 0.5
\end{array} \right.,
\]

\[
C(\nu_i) = \left\{ \begin{array}{ll}
0.25 & \text{for } \nu_i \leq 0.5 \\
0.5 & \text{for } \nu_i > 0.5.
\end{array} \right.
\]

Here \( <\ldots> \) denotes the standard inner (scalar) product in \( \mathbb{R}^m \) and \( \rho_{A_i} \) is the spectral radius of the Jacobi matrix \( \partial F / \partial W \) at the point \( W_i \) (for the case of the Euler equations, \( \rho_{A_i} = |u_i| + c_i \) where \( u_i \) is the local flow speed and \( c_i \) is the local sound speed).

### 3 Implicit finite volume method for 2D inviscid flows

The numerical solution is again obtained by the finite volume approach: The domain \( \Omega \) is approximated by a polygonal domain \( \Omega_h \) and this polygonal domain is divided into \( m \) polygonal convex cells\(^3\) \( C_i \) possess the following property:

\[
\Omega_h = \bigcup_{i=1}^{m} C_i \text{ and } C_i \cap C_j = \emptyset \text{ for } i \neq j.
\]

\(^3\)Since the structured grid can be viewed from the mathematical point of view as a special case of the unstructured grid, we present only the scheme for unstructured meshes.
Figure 1 shows a sample of such a domain divided into 13 triangular, quadrilateral, and pentagonal cells. Let $m_i$ denote the number of cells adjacent to $C_i$ (i.e. number of cells that share an edge with the cell $C_i$) and let the set $N_i = \{i_1, i_2, ..., i_{m_i}\}$ contain their indices (see Fig. 1 where $m_i = 5$). Next, let us denote by $B_i^{\text{inlet}}$ the set of edges shared by the cell $C_i$ and the inlet boundary $\Gamma_i^{\text{inlet}}$ of $\Omega_i$; similarly for $B_i^{\text{outlet}}$ and $B_i^{\text{wall}}$.

The basic finite volume scheme is obtained in the usual way: integrating the conservation law in a cell $C_i$, applying Green’s theorem and approximating the integral over the boundary of $C_i$ by the numerical flux functions. The scheme is then

$$W_i^{n+1} = W_i^n - \Delta t R^1(W^n)_i.$$  \hfill (39)

Here $W_i^n$ stands for the approximation of the solution in the cell $C_i$ at a time $t = n\Delta t$ and $R^1(W^n)_i$ is the component of the residual vector computed as

$$R^1(W^n)_i = \frac{1}{V(C_i)} \left[ \sum_{j \in N_i} H(W_i^n, W_j^n, \bar{S}_{i,j}) + \sum_b \sum_{e \in B_i^b} H^b(W_i^n, \bar{S}_e) \right].$$ \hfill (40)

The superscript 1 denotes the first order approximation, $\mu(C_i)$ is the volume of the cell $C_i$, $\bar{S}_{i,j}$ denotes the outer normal vector to the common edge between $C_i$ and $C_j$, the function $H$ is the numerical flux, $b$ denotes the type of the boundary conditions and belongs to the set $b \in \{\text{inlet, outlet, wall}\}$, $H^b$ is the numerical flux through the boundary and $\bar{S}_e$ denotes the outer normal vector to the boundary edge $e$. Both vectors $\bar{S}_{i,j}$ and $\bar{S}_e$ have the length equal to the length of the corresponding edge. The numerical flux $H(W_i^n, W_j^n, \bar{S}_{i,j})$ in the previous formula is the numerical approximation of the integral of the physical flux function over the common edge $e_{i,j}$ shared between $C_i$ and $C_j$:

$$H(W_i^n, W_j^n, \bar{S}_{i,j}) \approx \int_{e_{i,j}} \left( \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (e+p)u \end{pmatrix} \right) \cdot \begin{pmatrix} n_x \\ \rho v \\ \rho v^2 + p \\ (e+p)v \end{pmatrix} \left( \begin{array}{c} n_x \\ n_y \end{array} \right) dS$$ \hfill (41)

where $n_x$ and $n_y$ are the components of the unit normal vector to the edge $e_{i,j}$ oriented as the outer normal for the cell $C_i$. Analogous, $H^b(W_i^n, \bar{S}_e)$ is the approximation of the flux through the edge on the boundary.

### 3.1 First order implicit scheme

As a building block for the implicit scheme we choose the first order finite volume scheme based on Osher’s flux and the related approximate Riemann solver (see [12]). The advantage of the Osher flux is that one can evaluate simply the Jacobians of the numerical flux function which are needed for the implicit scheme.

The usual explicit first order scheme is then

$$W_i^{n+1} = W_i^n - \Delta t R^1(W^n)$$ \hfill (42)

and the implicit scheme is obtained from the explicit version (39) by replacing $R^1(W^n)$ by $R^1(W^{n+1})$:

$$W_i^{n+1} = W_i^n - \Delta t R^1(W^{n+1})_i.$$ \hfill (43)

The operator $R$ is nonlinear, so we can’t solve this equation directly. Therefore we linearize the equation at the point $W^n$:

$$W^{n+1} = W^n - \Delta t \left( R^1(W^n) + \frac{\partial R^1}{\partial W} (W^{n+1} - W^n) \right),$$ \hfill (44)

hence,

$$\left( \frac{I}{\Delta t} + \frac{\partial R^1}{\partial W} \right) (W^{n+1} - W^n) = -R^1(W^n).$$ \hfill (45)

The matrix $\frac{\partial R^1}{\partial W}$ is evaluated at the point $W^n$ using the expressions for the Jacobian matrices of Osher’s flux functions $\frac{\partial H(W_i, W_j, S)}{\partial W_i}$ and $\frac{\partial H(W_i, W_j, S)}{\partial W_j}$ and the appropriate Jacobian matrices of boundary fluxes.

The resulting system of linearized equations is solved using a GMRES method preconditioned with the ILU decomposition.
3.2 Second order semi-implicit scheme

In order to improve the accuracy of the basic first order scheme we use a piecewise linear reconstruction of the solution (for details see [7]). The second order semi-implicit scheme uses the high order residual $R^2$ on the right hand side (explicit part) and the matrix on the left hand side is computed from the low order residual:

$$\left(\frac{1}{\Delta t} + \frac{\partial R^1}{\partial W}\right)(W^{n+1} - W^n) = -R^2(W^n).$$

The second order residual $R^2$ is computed in the following way: we compute first the approximation of the gradient of the solution $\text{grad}W^n$ in each cell and then using this gradient we define the second order residual vector by

$$W^L_{i,j} = W^n_i + (\tilde{x}_{i,j} - \bar{x}_i) \cdot \text{grad}W^n_i,$$

$$W^R_{i,j} = W^n_j + (\tilde{x}_{j,i} - \bar{x}_j) \cdot \text{grad}W^n_j,$$

$$R^2(W^n)_i = \frac{1}{m(C_i)} \left[ \sum_{j \in N_i} H(W^L_{i,j}, W^R_{i,j}, S_{i,j}) + \sum_b \sum_{e \in B^i_b} H^b(W^L_{i,j}, S_e) \right],$$

where $\bar{x}_i$ is the center of gravity of the cell $i$ and $\tilde{x}_{i,j}$ is the center of the common edge between the cells $i$ and $j$. The numerical fluxes $H$ and $H^b$ are computed in the same way as for the first order scheme (i.e. using Osher’s Riemann solver) but instead of $W_i$ and $W_j$ we use the interpolated values $W^R_{i,j}$ and $W^L_{i,j}$.

4 2D transonic inviscid flow through a channel and a turbine cascade

4.1 Transonic flow through the 2D test channel with a bump

As a first test case we choose the transonic flow through the two-dimensional test channel with a bump, i.e. the so-called Ron-Ho-Ni channel. This is a well-known test case and it was solved by many researchers. See for example [3], [4].

Figure 2: Distribution of the Mach number along the lower and upper walls for the 2D channel.

![Figure 2](image)

(a) TVD MacCormack scheme

(b) Implicit scheme

Figure 2: Distribution of the Mach number along the lower and upper walls for the 2D channel.

We use the structured mesh with $120 \times 30$ quadrilateral cells for the MacCormack scheme and an unstructured triangular mesh with 4424 triangles (with refinement in the vicinity of the shock wave) for the implicit scheme. At the inlet ($x = -1$) we prescribe the stagnation pressure $p_0 = 1$, the stagnation density $\rho_0 = 1$ and the inlet angle $\alpha_1 = 0$. At the outlet we keep the pressure $p_2 = 0.737$. The upper ($y = 1$) and lower part are solid walls.

Figure 2(a) shows the distribution of the Mach number along the upper and the lower wall after 30000 iterations of two above mentioned variants of the MacCormack scheme with $CFL = 0.5$ while figure 2(b) shows the results obtained by the implicit scheme.

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The evaluation of the gradients is described in detail [7].
We can see that the results obtained by the full TVD MacCormack scheme are quite good. Causon’s simplified scheme uses too much artificial dissipation.

4.2 Transonic flow through the 2D turbine cascade SE 1050

Next we solve the transonic flow through the 2D turbine cascade SE 1050 given by Škoda Plzeň. Figure 3 shows the results of the interferometric measurement obtained for the inlet Mach number $M_1 = 0.395$. One can see the characteristic structure of the shock waves emitted from the outlet edge and the reflected shock waves. Moreover, one can notice the recompression zone on the suction side of the blade. Our computation was performed on a structured mesh with 200 $\times$ 40 quadrilateral cells for the full TVD MacCormack scheme (see Fig. 4(a)) and on an unstructured mesh with 7892 triangles for the implicit scheme (Fig. 4(b)).

4.3 Laminar viscous flow through a 2D turbine cascade

Next we solve the transonic viscous flow through the DCA 8% cascade. We consider the flow with the non-dimensional viscosity $\mu = 10^{-4}$ which gives, for the inlet conditions $p_0 = 1$, $\rho_0 = 1$, $\alpha_1 = 2^\circ$ and outlet pressure $p_2 = 0.48$, the value of the Reynolds number $Re = 6450$, inlet Mach number $M_1 = 0.76$ and outlet Mach number $M_2 = 1.03$.

We use a simple structured mesh with 90 $\times$ 50 quadrilateral cells refined in the vicinity of profiles. Figure 5 shows the results obtained by an improved version of Causon’s scheme after 50000 and 50200 iterations. We can see that the solution is non-stationary (see the changes in the shape of the wake). Let us mention that

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\textsuperscript{5}The so called modified scheme which we published for example in [8].
these flow conditions (it means relatively low Reynolds number and high Mach number) are not interesting for practical applications. We have done this computation in order to show that the effects of artificial viscosity can be very important for a viscous flow calculation. As a matter of fact, a similar case was formerly solved by M. Huněk and K. Kozel [5] using an implicit residual averaging method. Their method gave stationary results whereas our computation leads to a non-stationary solution. This is probably due to the fact, that the residual smoothing method uses too much artificial viscosity.

5 3D transonic flow through a turbine cascade

The three-dimensional Causon’s scheme and its improved variant are used for the computation of the transonic flow through the stator stage of the real 3D turbine given by Skoda Plzeň company.

At the inlet we prescribe the stagnation pressure $p_0(r) = 0.38274$, stagnation density $\rho_0(r) = 1$. The direction of the velocity at the inlet is given by two angles $\alpha_1(r)$ and $\mu_1(r)$.

We use a structured mesh with $90 \times 24 \times 17$ hexahedral cells.

![Figure 5: Isolines of Mach number for the non-stationary laminar transonic flow through the DCA 8% cascade.](image)

![Figure 6: Distribution of Mach number for the sections $k = \text{const}$.](image)
(a) Mach number distribution on the pressure side of the blade

(b) Mach number distribution on the suction side of the blade

Figure 7: Distribution of Mach number on the blade

References


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