Understanding large-scale atmospheric and oceanic flows with layered rotating shallow water models

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Plan

General problematics

Importance of moisture in the atmosphere: obvious. Influences large-scale dynamics via the latent heat release, due to condensation and precipitation.
Atmospheric circulation modeling: equation of state of the moist air extremely complex. Discretization/averaging: problematic.
Current parametrizations of precipitations and latent heat release:
relaxation to the equilibrium (saturation) profile of humidity $\Rightarrow$ threshold effect $\Rightarrow$ essential nonlinearity
Consequences: no linear limit; linear thinking: modal decomposition, linear stability analysis, etc impossible $\Rightarrow$ problems in quantifying predictability of moist - convective dynamical systems.

Aims and method I

Aim:

Understanding the influence of condensation and latent heat release upon large-scale dynamical processes

Reminder:

- Simplest model for large-scale motions: rotating shallow water.
- Link with primitive equations: vertical averaging
- Baroclinic effects: 2 (or more) layers.

Problem with this approach for moist air: averaging of essentially nonlinear equation of state.

Aims and method II

Our approach

- Combine (standard) vertical averaging of primitive equations between the isobaric surfaces with that of Lagrangian conservation of moist enthalpy
- Allow for convective fluxes (extra vertical velocity) across the isobars
- Link these fluxes to condensation
- Use relaxation parametrization in terms of bulk moisture in the layer for the condensation/precipitation

Aims and method III

Advantages:

- Simplicity, qualitative analysis of basic phenomena straightforward
- Fully nonlinear in the hydrodynamic sector
- Well-adapted for studying discontinuities, in particular precipitation fronts
- Efficient numerical tools available (finite-volume codes for shallow water)
- Various limits giving known models
- Inclusion of topography (gentle or steep) straightforward

Primitive equations in pseudo-height coordinates

$$
\begin{aligned}
&\frac{d}{dt}\boldsymbol{v} + f\boldsymbol{k} \times \boldsymbol{v} = -\nabla\phi \\
&\frac{d}{dt}\theta = 0 \\
&\nabla \cdot \boldsymbol{v} + \partial_z w = 0 \\
&\partial_z \phi = g\frac{\theta}{\theta_0}
\end{aligned}
$$

$\boldsymbol{v} = (u, v)$ and w - horizontal and vertical velocities, $\frac{d}{dt} = \partial_t + \boldsymbol{v} \cdot \nabla + w\partial_z$, f - Coriolis parameter, θ - potential temperature, ϕ - geopotential.

Moisture and moist enthalpy

Condensation turned off: conservation of specific humidity of the air parcel:

$$\frac{d}{dt}q = 0.$$

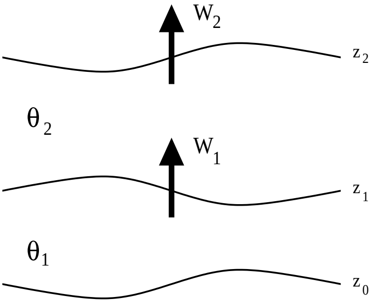
Condensation turned on: θ and q equations acquire source and sink. Yet the moist enthalpy $\theta + \frac{L}{c_p}q$, where L - latent heat release, c_p - specific heat, is conserved for any air parcel on isobaric surfaces:

$$\frac{d}{dt}\left(\theta + \frac{L}{c_p}q\right) = 0,$$

Vertical averaging with convective fluxes

3 material surfaces:

$$w_0 = \frac{dz_0}{dt}, \quad w_1 = \frac{dz_1}{dt} + W_1, \quad w_2 = \frac{dz_2}{dt} + W_2.$$



Mean-field + constant mean $\theta \rightarrow$

Averaged momentum and mass conservation equations:

$$\begin{cases} \partial_t \boldsymbol{v}_1 + (\boldsymbol{v}_1 \cdot \nabla)\boldsymbol{v}_1 + f\boldsymbol{k} \times \boldsymbol{v}_1 = -\nabla\phi(z_1) + g\frac{\theta_1}{\theta_0}\nabla z_1, \\ \partial_t \boldsymbol{v}_2 + (\boldsymbol{v}_2 \cdot \nabla)\boldsymbol{v}_2 + f\boldsymbol{k} \times \boldsymbol{v}_2 = -\nabla\phi(z_2) + g\frac{\theta_2}{\theta_0}\nabla z_2 + \frac{\boldsymbol{v}_1 - \boldsymbol{v}_2}{h_2} W_1, \end{cases}$$

$$\begin{cases} \partial_t h_1 + \nabla \cdot (h_1 \boldsymbol{v}_1) = -W_1, \\ \partial_t h_2 + \nabla \cdot (h_2 \boldsymbol{v}_2) = +W_1 - W_2, \end{cases}$$

Linking convective fluxes to precipitation I

Bulk humidity: $Q_i = \int_{z_{i-1}}^{z_i} q dz$. Precipitation sink:

$$\partial_t Q_i + \nabla \cdot (Q_i \boldsymbol{v}_i) = -P_i.$$

In precipitating regions ($P_i > 0$), moisture is saturated $q(z_i) = q^s(z_i)$ and the temperature of the air-mass $W_i dtdxdy$ convected due to the latent heat release $\theta(z_i) + \frac{L}{c_p} q^s(z_i)$, is the one of the upper layer: θ_{i+1}. We assume "dry" stable background stratification:

$$\theta_{i+1} = \theta(z_i) + \frac{L}{c_p} q(z_i) \approx \theta_i + \frac{L}{c_p} q(z_i) > \theta_i,$$

with constant $\theta(z_i)$ and $q(z_i)$.

Integrating the moist enthalpy we get

$$W_i = \beta_i P_i$$

with a positive-definite coefficient

$$\beta_i = \frac{L}{c_p(\theta_{i+1} - \theta_i)} \approx \frac{1}{q(z_i)} > 0.$$

Last step: relaxation formula with relaxation time τ.

$$P_i = \frac{Q_i - Q_i^s}{\tau} H(Q_i - Q_i^s)$$

where $H(.)$ is the Heaviside (step) function.

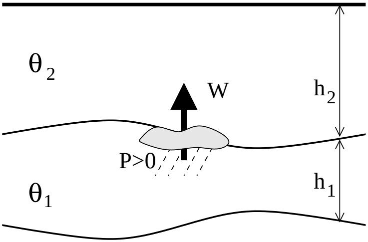
2-layer model with a dry upper layer

Vertical boundary conditions: upper surface isobaric $z_2 = \mathrm{const}$, geopotential at the bottom constant (ground) $\phi(z_0) = \mathrm{const}$, $Q_2 = 0$, $Q_1 = Q$:

$$\begin{cases} \partial_t \mathbf{v}_1 + (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1 + f\mathbf{k} \times \mathbf{v}_1 = -g\nabla(h_1 + h_2), \\ \partial_t \mathbf{v}_2 + (\mathbf{v}_2 \cdot \nabla)\mathbf{v}_2 + f\mathbf{k} \times \mathbf{v}_2 = -g\nabla(h_1 + \alpha h_2) + \frac{\mathbf{v}_1 - \mathbf{v}_2}{h_2}\beta P, \\ \partial_t h_1 + \nabla \cdot (h_1 \mathbf{v}_1) = -\beta P, \\ \partial_t h_2 + \nabla \cdot (h_2 \mathbf{v}_2) = +\beta P, \\ \partial_t Q + \nabla \cdot (Q\mathbf{v}_1) = -P, \quad P = \frac{Q - Q^s}{\tau} H(Q - Q^s) \end{cases}$$

$\alpha = \frac{\theta_2}{\theta_1}$ - stratification parameter.

Sketch of the model



Immediate relaxation limit

$\tau \to 0, \quad \Rightarrow \quad P = -Q^s \nabla \cdot \boldsymbol{v}_1$ (Gill, 1982), and

$$\partial_t \boldsymbol{v}_1 + (\boldsymbol{v}_1 \cdot \nabla)\boldsymbol{v}_1 + f\boldsymbol{k} \times \boldsymbol{v}_1 = -g\nabla(h_1 + h_2),$$

$$\partial_t \boldsymbol{v}_2 + (\boldsymbol{v}_2 \cdot \nabla)\boldsymbol{v}_2 + f\boldsymbol{k} \times \boldsymbol{v}_2 = -g\nabla(h_1 + \alpha h_2)$$

$$-\frac{\boldsymbol{v}_1 - \boldsymbol{v}_2}{h_2}\beta Q^s \nabla \cdot \boldsymbol{v}_1,$$

$$\partial_t h_1 + \nabla \cdot (h_1 \boldsymbol{v}_1) = +\beta Q^s \nabla \cdot \boldsymbol{v}_1,$$

$$\partial_t h_2 + \nabla \cdot (h_2 \boldsymbol{v}_2) = -\beta Q^s \nabla \cdot \boldsymbol{v}_1,$$

humidity staying at the saturation value: $Q = Q^s$.

Baroclinic reduction

Rewriting the model in terms of baroclinic and barotropic velocities:

$$\boldsymbol{v}^{bt} = \frac{h_1 \boldsymbol{v}_1 + h_2 \boldsymbol{v}_2}{h_1 + h_2}, \;\; \boldsymbol{v}^{bc} = \boldsymbol{v}_1 - \boldsymbol{v}_2,$$

and linearizing in the hydrodynamic sector gives:

$$\begin{cases} \partial_t \boldsymbol{v}^{bc} + f \boldsymbol{k} \times \boldsymbol{v}^{bc} = -g_e \nabla \eta, \\ \partial_t \eta + H_e \nabla \cdot \boldsymbol{v}^{bc} = -\beta P, \\ \partial_t Q + Q_e \nabla \cdot \boldsymbol{v}^{bc} = -P, \end{cases} ,$$

where $g_e = g(\alpha - 1)$, $Q_e = \frac{H_e}{H_1} Q^s$, η - perturbation of the interface, H_e - equivalent height.
Model first proposed by Gill (1982) and studied by Majda *et al* (2004, 2006, 2008).

Quasigeostrophic limit

In the small Rossby number limit on the β-plane Lapeyre & Held (2004) model follows:

$$\frac{d_1^{(0)}}{dt}(\nabla^2\psi_1 + y - \frac{\eta_1}{D_1}) = \frac{\beta P}{D_1},$$
$$\frac{d_2^{(0)}}{dt}(\nabla^2\psi_2 + y - \frac{\eta_2}{D_2}) = -\frac{\beta P}{D_2},$$

Here $\frac{d_i^{(0)}}{dt} = \partial_t + (\boldsymbol{v}_i^{(0)} \cdot \nabla)$, $\boldsymbol{k} \times \boldsymbol{v}_i^{(0)} = -\nabla\psi_i$, $D_i = \frac{H_i}{H_0}$, and $\psi_{1,2}$ (geostrophic streamfunctions) are related to the free-surface (η_2) and interface (η_1) perturbations as:

$$\psi_1 = \eta_1 + \eta_2, \quad \psi_2 = \eta_1 + \alpha\eta_2.$$

1-layer moist-convective RSW

In the limit $H_1/(H_1 + H_2) \to 0$ the reduced-gravity one-layer moist-convective shallow water follows (Bouchut, Lambaerts, Lapeyre & Zeitlin, 2009):

$$\begin{cases} \partial_t \boldsymbol{v}_1 + (\boldsymbol{v}_1 \cdot \nabla)\boldsymbol{v}_1 + f\boldsymbol{k} \times \boldsymbol{v}_1 = -\nabla\eta, \\ \partial_t \eta + \nabla \cdot \{\boldsymbol{v}_1 (1+\eta)\} = -\beta P, \\ \partial_t Q + \nabla \cdot (Q\boldsymbol{v}_1) = -P, \end{cases}$$

(Nondimensional equations, η - free-surface perturbation)

Horizontal momentum

$$(\partial_t + \boldsymbol{v}_1 \cdot \nabla)(\boldsymbol{v}_1 h_1) + \boldsymbol{v}_1 h_1 \nabla \cdot \boldsymbol{v}_1 + f\boldsymbol{k} \times (\boldsymbol{v}_1 h_1) = -g\nabla \frac{h_1^2}{2} - gh_1 \nabla h_2 - \boldsymbol{v}_1 \beta P,$$

$$(\partial_t + \boldsymbol{v}_2 \cdot \nabla)(\boldsymbol{v}_2 h_2) + \boldsymbol{v}_2 h_2 \nabla \cdot \boldsymbol{v}_2 + f\boldsymbol{k} \times (\boldsymbol{v}_2 h_2) = -\alpha g\nabla \frac{h_2^2}{2} - gh_2 \nabla h_1 + \boldsymbol{v}_1 \beta P,$$

Red: moist convection drag. Total momentum: $\boldsymbol{v}_1 h_1 + \boldsymbol{v}_2 h_2$ is not affected by convection.

Energy

Energy densities of the layers:

$$\begin{cases} \mathrm{e}_1 = h_1 \frac{\boldsymbol{v}_1^2}{2} + g\frac{h_1^2}{2}, \\ \mathrm{e}_2 = h_2 \frac{\boldsymbol{v}_2^2}{2} + gh_1 h_2 + \alpha g \frac{h_2^2}{2}, \end{cases}$$

For the total energy $E = \int dxdy(\mathrm{e}_1 + \mathrm{e}_2)$ we get:

$$\partial_t \mathrm{E} = -\int d\boldsymbol{x}\ \beta P \left(gh_2(1-\alpha) + \frac{(\boldsymbol{v}_1 - \boldsymbol{v}_2)^2}{2} \right).$$

1st term: production of PE (for stable stratification); 2nd term destruction of KE.

Potential vorticity

$$(\partial_t + \boldsymbol{v}_1 \cdot \nabla)\frac{\zeta_1 + f}{h_1} = \frac{\zeta_1 + f}{h_1^2}\beta P,$$
$$(\partial_t + \boldsymbol{v}_2 \cdot \nabla)\frac{\zeta_2 + f}{h_2} = -\frac{\zeta_2 + f}{h_2^2}\beta P + \frac{\boldsymbol{k}}{h_2} \cdot \left\{\nabla \times \left(\frac{\boldsymbol{v}_1 - \boldsymbol{v}_2}{h_2}\beta P\right)\right\},$$

where $\zeta_i = \boldsymbol{k} \cdot (\nabla \times \boldsymbol{v}_i) = \partial_x v_i - \partial_y u_i$ ($i = 1, 2$)- relative vorticity.

PV in each layer is not a Lagrangian invariant in precipitating regions.

Moist enthalpy and moist PV

Moist enthalpy in the lower layer: $m_1 = h_1 - \beta Q$ and is always locally conserved:

$$\partial_t m_1 + \nabla \cdot (m_1 \boldsymbol{v}_1) = 0.$$

Conservation of the moist enthalpy in the lower layer allows to derive a new Lagrangian invariant, the moist PV:

$$(\partial_t + \boldsymbol{v}_1 \cdot \nabla) \frac{\zeta_1 + f}{m_1} = 0.$$

Quasilinear form and characteristic equations

1-d reduction: $\partial_y(...) = 0, \quad \Rightarrow$ quasilinear system:

$$\partial_t \boldsymbol{f} + \mathbf{A}(\boldsymbol{f})\partial_x \boldsymbol{f} = \boldsymbol{b}(\boldsymbol{f}).$$

Characteristic equation: $\det(\mathbf{A} - cI) = 0$

- "Dry" characteristic equation

$$\mathcal{F}(c) = \left\{(u_1 - c)^2 - gh_1\right\}\left\{(u_2 - c)^2 - \alpha gh_2\right\} - gh_1 gh_2 = 0,$$

- "Moist" characteristic equation ($\tau \to 0$)

$$\mathcal{F}^m(c) = \mathcal{F}(c) + ((u_1 - u_2)^2 - (\alpha - 1)gh_2)g\beta Q^s = 0.$$

Characteristic velocities about the rest state

- "Dry" characteristics:

$$C_{\pm} = g(H_1 + \alpha H_2)\frac{1 \pm \sqrt{\Delta}}{2},$$

- "Moist" characteristics:

$$C^m_{\pm} = g(H_1 + \alpha H_2)\frac{1 \pm \sqrt{\Delta^m}}{2}.$$

Here $C = c^2$ and

$$\Delta = 1 - \frac{4H_1H_2(\alpha - 1)}{(H_1 + \alpha H_2)^2} = \frac{(H_1 - \alpha H_2)^2 + 4H_1H_2}{(H_1 + \alpha H_2)^2}.$$

$$\Delta^m = \Delta + \frac{4(\alpha - 1)\beta Q^s H_2}{(H_1 + \alpha H_2)^2}.$$

Moist vs dry characteristic velocities

c^m is real for positive moist enthalpy of the lower layer in the state of rest : $M_1 = H_1 - \beta Q^s > 0$, and

$$C_-^m < C_- < \frac{g(H_1 + \alpha H_2)}{2} < C_+ < C_+^m,$$

for $0 < M_1 < H_1 \Rightarrow$ moist internal (mainly baroclinic) mode propagates slower than the dry one, consistent with observations.

Discontinuities in dependent variables (no rotation)

Rankine-Hugoniot (RH) conditions (immediate relaxation):

$$\begin{cases} -s[v_1 h_1 + v_2 h_2] + [u_1 v_1 h_1 + u_2 v_2 h_2] = 0, \\ -s[m_1] + [m_1 u_1] = 0, \\ -s[h_2] + [h_2 u_2 + \beta Q^s u_1] = 0. \end{cases}$$

s - propagation speed of the discontinuity.
Remark: mass conservation $\rightarrow$ moist enthalpy conservation in the lower layer.
Due to $\lim_{x_s \rightarrow a} \lim_{b \rightarrow x_s} \int_a^b P = 0$, P does not enter RH conditions for u, v, h.

Discontinuities in derivatives

RH conditions linearized about the rest state:

$$\begin{cases} (s^2 - C_+)(s^2 - C_-)[\partial_x u_1] = -(\alpha - 1)gH_2 g\beta[P], \\ (s^2 - C_+^m)(s^2 - C_-^m)[\partial_x u_1] = -s(\alpha - 1)gH_2 g\beta[\partial_x Q]. \end{cases}$$

For a configuration where it rains at the right side of the discontinuity, $P_- = 0$ and $P_+ = -Q^s \partial_x u_{1+} > 0$, there exist five types of precipitation fronts:

Precipitation fronts

1. the dry external fronts, $\sqrt{C_+} < s < \sqrt{C_+^m}$,
2. the dry internal subsonic fronts, $\sqrt{C_-^m} < s < \sqrt{C_-}$,
3. the moist internal subsonic fronts, $-\sqrt{C_-^m} < s < 0$,
4. the moist internal supersonic fronts, $-\sqrt{C_+} < s < -\sqrt{C_-}$,
5. the moist external fronts, $s < -\sqrt{C_+^m}$.

This result confirms previous studies within a linear baroclinic model (Frierson *et al*, 2004).

Wave scattering on a moisture front: setting

Localized internal simple wave centred at $x_P = 2$ and moving eastward:

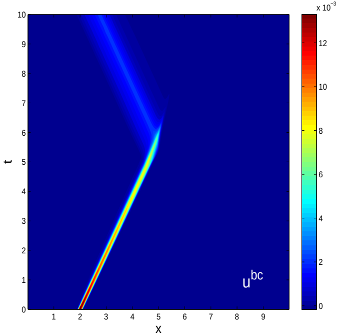
$$u_1(x,0) = \begin{cases} \sigma(x-x_P)^2 + U_0 & \text{if} \quad -\sqrt{\frac{U_0}{\sigma}} \leq x - x_P \leq \sqrt{\frac{U_0}{\sigma}}, \\ 0 & \text{otherwise}, U_0 = 0.01, \sigma = -1 \end{cases} \tag{1}$$

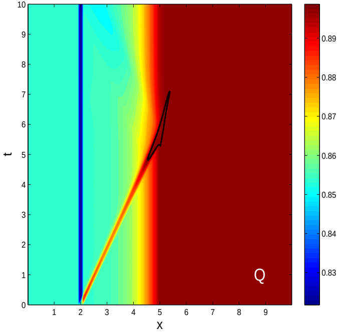
Stationary moisture front at $x_M = 5$, saturated air at the east, unsaturated at the west:

$$Q(x,0) = Q^s\{1 + q_0 \tanh(x - x_M) H(-x + x_M)\}, q_0 = 0.05. \tag{2}$$

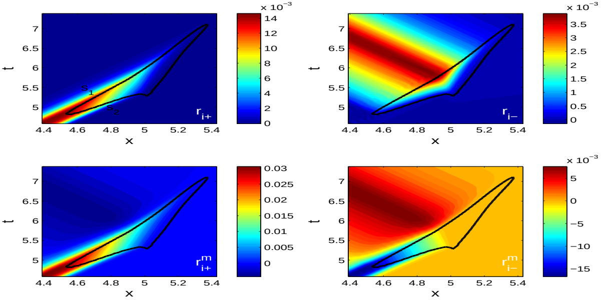
Strong downflow convergence in the lower layer $\rightarrow P > 0$ near the moisture front.

Wave scattering on a moisture front: baroclinic velocity and moisture



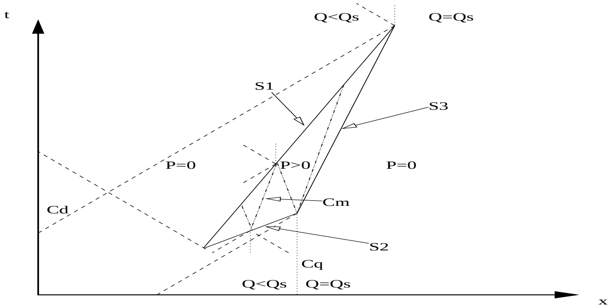


Wave scattering on a moisture front: condensation zone I



Dry and moist internal Riemann invariants. $s_{1,2}$ - precipitation fronts (dry subsonic and moist supersonic).

Characteristics and fronts in the condensation zone



Evaporation and its parametrizations

In the presence of evaporation source E

$$\partial_t Q + \nabla \cdot (Q\boldsymbol{v}_1) = E - P$$

Hence:

$$\partial_t m_1 + \nabla \cdot (m_1 \boldsymbol{v}_1) = -\beta E$$

Simple parametrizations of E (may be combined):

- Relaxational: $E = \frac{\hat{Q}-Q}{\tau_E} H(m_1)$, where $\hat{Q}$ - equilibrium value.
- Dynamic: $E = \alpha_E |\mathbf{v}_1| H(m_1)$

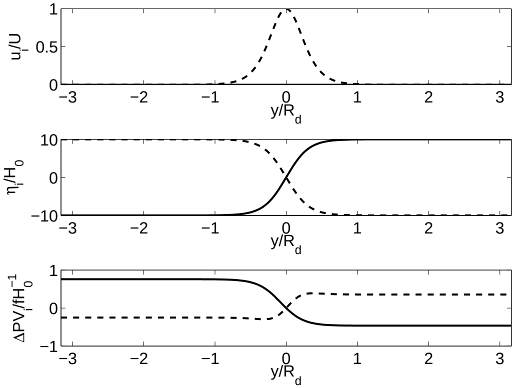
m_1 should stay positive (plays a role of static stability)

Baroclinic Bickley jet

Geostrophically balanced upper-layer jet on the f-plane. non-dimensional profiles of velocity and thikness perturbations:

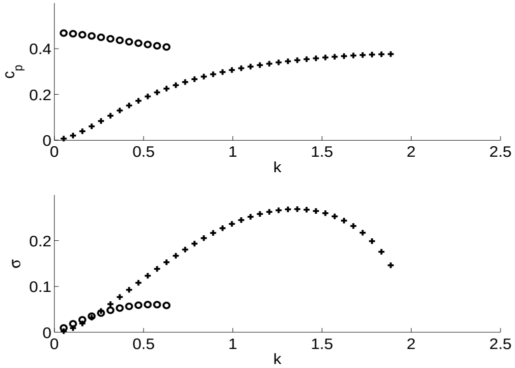
$$\bar{u}_1 = 0, \quad \bar{\eta}_1 = \frac{1}{\alpha - 1}\tanh(y),$$
$$\bar{u}_2 = \mathrm{sech}^2(y), \quad \bar{\eta}_2 = \frac{-1}{\alpha - 1}\tanh(y).$$

No deviation of the free surface: $\bar{\eta}_1 + \bar{\eta}_2 = 0$.
Parameters: $Ro = 0.1$, $Bu = 10$ - typical for atmospheric jets.



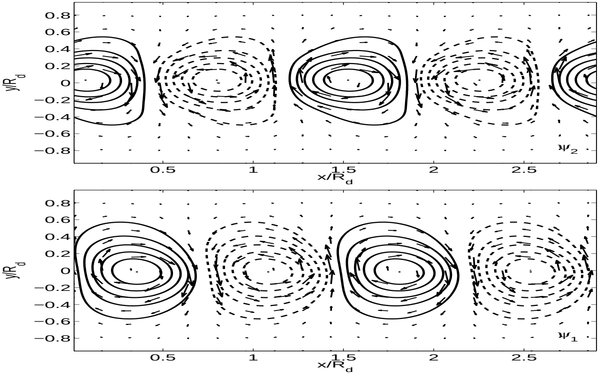
Bickley jet: zonal velocity $\bar{u}_i$, thickness deviation $\bar{\eta}_i$ and PV anomaly. Lower (upper) layer: solid black (dashed gray).

Linear stability diagram



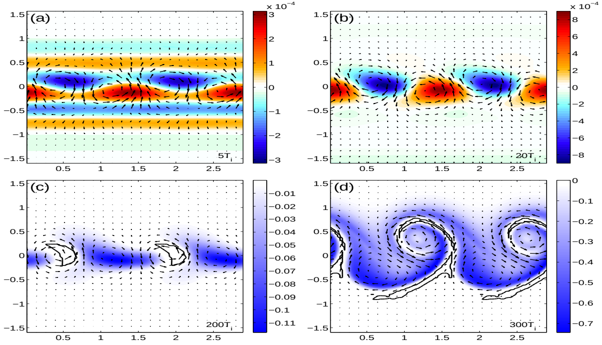
Phase velocity (top) and growth rate (bottom)

The most unstable mode



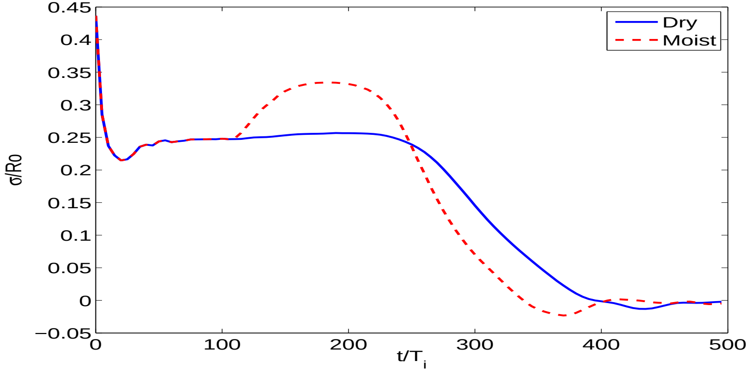
Most unstable mode of the upper-layer Bickley jet. Upper(top) and lower (bottom) layer- geostrophic streamfunctions and velocity (arrows) fields.

Early stages: evolution of moisture



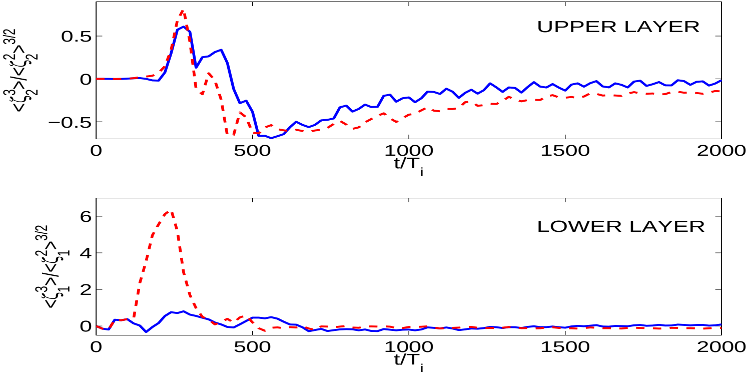
Evolution of the moisture anomaly $Q - Q_0$ with superimposed lower-layer velocity. Black contour: condensation zones.

Early stages: growth rates



Red: moist, blue: dry simulations. $\Rightarrow$
Transient increase in the growth rate due to condensation.

Cyclone-anticyclone asymmetry



Skewness of relative vorticity. Red: moist, blue: dry simulations.

How condensation enhances cyclones: 1-layer model

For $Ro \to 0$ and $Bu \sim O(1)$, close to saturation $\psi \sim \tilde{q} << 1$:

$$(\partial_t + \boldsymbol{v}^{(0)} \cdot \nabla) \left[\nabla^2 \psi - \psi \right] = \beta P, \qquad (3)$$

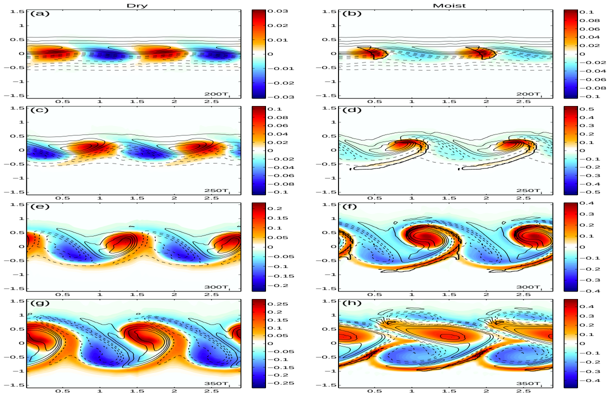
$$(\partial_t + \boldsymbol{v}^{(0)} \cdot \nabla) \left[\tilde{q} - Q_s \nabla^2 \psi \right] = -P, \qquad (4)$$

$\boldsymbol{v}^{(0)} = (-\partial_y \psi, \partial_x \psi)$ - geostrophic velocity, $\psi = \bar{\eta} + \eta$, and $\tilde{q}$ is moisture anomaly with respect to Q_s.
$\Rightarrow$ PV of the fluid columns which pass through the precipitating regions increases. For $\tau \to 0$ $\tilde{q} \approx 0$, and:

$$Q_s(\partial_t + \boldsymbol{v}^{(0)} \cdot \nabla) \left[\nabla^2 \psi \right] \approx P_{\tau \to 0} > 0, \qquad (5)$$

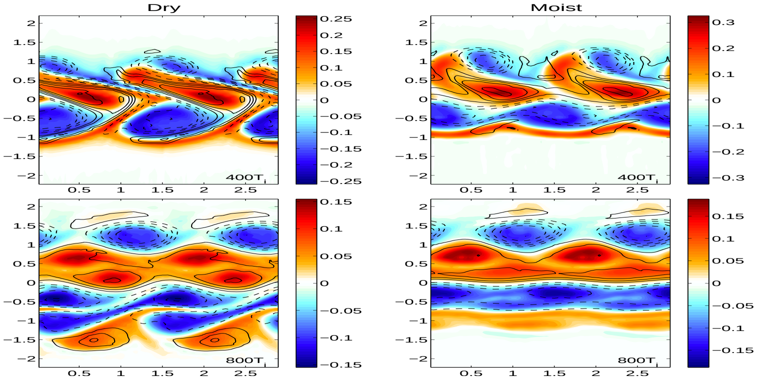
$\Rightarrow$ increase of geostrophic vorticity in the precipitation regions.

Dry vs moist simulations: evolution of relative vorticity

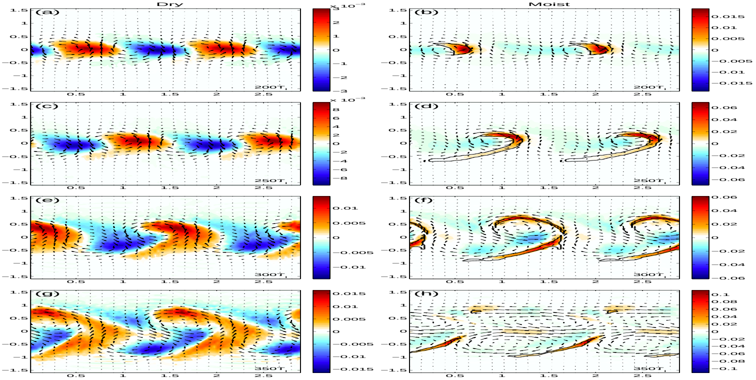


Lower layer: colors, upper layer: contours. Condensation: solid black.

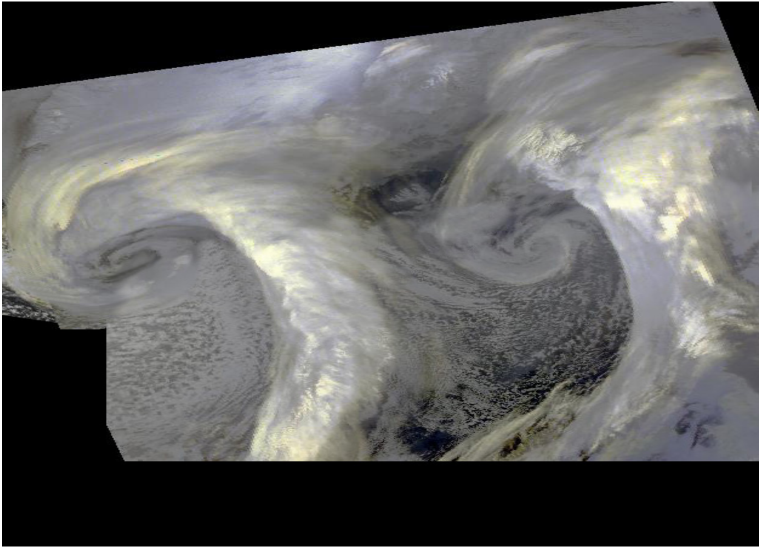
Dry vs moist simulations: formation of secondary zonal jets at late stages



Unbalanced (aheostrophic) motions: baroclinic divergence



Moist baroclinic instability in Nature



Conclusions 1

The model

- Physically and mathematically consistent
- Simple, physics transparent
- Efficient high-resolution numerical schemes available
- Benchmarks: good

Conclusions 2

Moist vs dry baroclinic instability

- local enhancement of the growth rate of the moist-convective instability at the precipitation onset,
- significant increase in intensity of ageostrophic motions during the evolution of the moist instability,
- substantial cyclone - anticyclone asymmetry, which develops due to the moist convection effects.
- substantial differences in the structure of zonal jets resulting at the late stage of saturation.

References

Presentation based on:

- Lambaerts J.; Lapeyre G.; Zeitlin V. and Bouchut F. "Simplified two-layer models of precipitating atmosphere and their properties" *Phys. Fluids* **23**, 046603, 2011.
- Lambaerts J.; Lapeyre G. and Zeitlin V. "Moist versus Dry Baroclinic Instability in a Simplified Two-Layer Atmospheric Model with Condensation and Latent Heat Release" *J. Atmos. Sci.* **69**, 1405-1426, 2012.