Understanding large-scale atmospheric and oceanic flows with layered rotating shallow water models

V. Zeitlin,

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Plan

Well-balanced finite-volume numerical schemes for 1-layer RSW

- Conceptual example : shallow water with topography
- Hydrostatic reconstruction
- Rotation as apparent topogrpahy
- Going second order
- Going two-dimensional

Numerical tests in 1.5 dimensions

- Adjustment of an unbalanced jet
- Simulations vs theory

Numerical tests in 2 dimensions

- Geostrophic adjustment of a monopolar perturbation
- (A)geostrophic adjustment of dipolar perturbations

Numerical scheme for 2-layer RSW

Literature

Reminder

RSW system :

- Equivalent to 2d barotropic gas dynamics (if no topography and rotation).
- Hyperbolic (except at resonant points (crossing of eigenvalues of the characteristic matrix).
- Rotation - stiff source.
- Weak solutions $\leftrightarrow$ Rankine-Hugoniot conditions. Selection : energy decrease across shocks (equivalent to entropy increase in gas dynamics).
- Natural numerical method : finite-volume, shock-capturing

1-dimensional SW with topography :

Equations in conservative form, where $Z(x)/g$ - topography :

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + (hu^2 + gh^2/2)_x + hZ_x = 0, \end{cases} \tag{1}$$

Convex entropy (energy) :

$$e = hu^2/2 + gh^2/2 + ghZ$$

with entropy flux $\left(e + gh^2/2\right) u$.

Numerical difficulties :

- keeping $h \geq 0$,
- maintaining steady states at rest ("well-balanced" property $u = 0$, $gh + Z =$ const
- treatment of drying $h \to 0$,
- satisfying a discrete entropy inequality.

First-order three-point finite-volume schemes

Discretization :

Grid $x_{i+1/2}$, $i \in \mathbb{Z}$, cells (finite volumes)
$C_i = (x_{i-1/2}, x_{i+1/2})$, centers $x_i = (x_{i-1/2} + x_{i+1/2})/2$,
lengths $\Delta x_i = x_{i+1/2} - x_{i-1/2}$.
Discrete data (U_i^n, Z_i), U_i^n – approximation of $U = (h, hu)$.

Evolution :

$$U_i^{n+1} - U_i^n + \frac{\Delta t}{\Delta x_i}(F_{i+1/2-} - F_{i-1/2+}) = 0, \qquad (2)$$

Z_i does not evolve,

$$F_{i+1/2-} = \mathcal{F}_l(U_i, U_{i+1}, \Delta Z_{i+1/2}),\ F_{i+1/2+} = \mathcal{F}_r(U_i, U_{i+1}, \Delta Z_{i+1/2}), \qquad (3)$$

with $\Delta Z_{i+1/2} = Z_{i+1} - Z_i$.

Consistency

Numerical fluxes $\mathcal{F}_l$ and $\mathcal{F}_r$ must satisfy two consistency properties.

- consistency with the conservative term :

$$\mathcal{F}_l(U, U, 0) = \mathcal{F}_r(U, U, 0) = F(U) \equiv (hu, hu^2 + gh^2/2), \tag{4}$$

- consistency with the source :

$$\mathcal{F}_r(U_l, U_r, \Delta Z) - \mathcal{F}_l(U_l, U_r, \Delta Z) = (0, -h\Delta Z) + o(\Delta Z), \tag{5}$$

as U_l, $U_r \to U$ and $\Delta Z \to 0$.

Well-balancing and mass conservation

A required global property is the conservation of mass,

$$\mathcal{F}_l^h(U_l, U_r, \Delta Z) = \mathcal{F}_r^h(U_l, U_r, \Delta Z) \equiv F^h(U_l, U_r, \Delta Z). \quad (6)$$

The property for the scheme to be well-balanced is that

$$F_{i+1/2-} = F(U_i) \text{ and } F_{i+1/2+} = F(U_{i+1})$$
$$\text{whenever } u_i = u_{i+1} = 0 \text{ and } gh_{i+1} - gh_i + \Delta Z_{i+1/2} = 0. \quad (7)$$

Hydrostatic reconstruction scheme (Audusse *et al*, 2003)

$$
\begin{aligned}
F_l(U_l, U_r, \Delta Z) &= \mathcal{F}(U_l^*, U_r^*) + \begin{pmatrix} 0 \\ \frac{g}{2}h_l^2 - \frac{g}{2}h_{l*}^2 \end{pmatrix}, \\
F_r(U_l, U_r, \Delta Z) &= \mathcal{F}(U_l^*, U_r^*) + \begin{pmatrix} 0 \\ \frac{g}{2}h_r^2 - \frac{g}{2}h_{r*}^2 \end{pmatrix},
\end{aligned}
\tag{8}
$$

where $U_l^* = (h_{l*}, h_{l*}u_l)$, $U_r^* = (h_{r*}, h_{r*}u_r)$, and

$$
\begin{aligned}
h_{l*} &= \max(0, h_l - \max(0, \Delta Z/g)), \\
h_{r*} &= \max(0, h_r - \max(0, -\Delta Z/g)).
\end{aligned}
$$

$\mathcal{F}$ is any entropy satisfying consistent numerical flux for the problem with $Z = cst$. Multiple choices for $\mathcal{F}$ in the literature - approximate Riemann solvers (Roe, HLL, HLLC,...).

Rotation as an apparent topography

1.5d shallow water with topography and Coriolis force

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + (hu^2 + gh^2/2)_x + hZ_x - fhv = 0, \\ (hv)_t + (huv)_x + fhu = 0, \end{cases} \tag{9}$$

where $Z = Z(x)$, $f = f(x)$. Solutions at rest are given by $u = 0$, $fv = (gh + Z)_x$. The trick is to identify the two first equations in (9) as (1) with a new topography $Z + B$, where $B_x = -fv$. As v depends on time while B should be time-independent, so take $B_x^n = -fv^n$ and solve (1) on the time interval (t_n, t_{n+1}) with topography $Z + B^n$.

Discretized 1.5d RSW

$$\begin{aligned} h_i^{n+1} - h_i^n + \frac{\Delta t}{\Delta x_i}(F^h_{i+1/2} - F^h_{i-1/2}) = 0, \\ h_i^{n+1}u_i^{n+1} - h_i^n u_i^n + \frac{\Delta t}{\Delta x_i}(F^{hu}_{i+1/2-} - F^{hu}_{i-1/2+}) = 0, \\ h_i^{n+1}v_i^{n+1} - h_i^n v_i^n + \frac{\Delta t}{\Delta x_i}(F^{hv}_{i+1/2-} - F^{hv}_{i-1/2+}) = 0, \end{aligned} \quad (10)$$

with

$$\begin{aligned} (F^h_{i+1/2}, F^{hu}_{i+1/2-}) = \mathcal{F}_l^{1d}(h_i, u_i, h_{i+1}, u_{i+1}, \Delta z_{i+1/2} + \Delta b^n_{i+1/2}), \\ (F^h_{i+1/2}, F^{hu}_{i+1/2+}) = \mathcal{F}_r^{1d}(h_i, u_i, h_{i+1}, u_{i+1}, \Delta z_{i+1/2} + \Delta b^n_{i+1/2}), \end{aligned} \quad (11)$$

$$\Delta b^n_{i+1/2} = -f_{i+1/2}\frac{v_i^n + v_{i+1}^n}{2}\Delta x_{i+1/2}/g. \quad (12)$$

and $\mathcal{F}_l^{1d}$ and $\mathcal{F}_r^{1d}$ - numerical hydrostatic reconstruction fluxes of the 1d shallow water.

Transverse momentum fluxes

A natural discretization associated to the equivalent conservation law (geostrophic momentum) $(h(v+\Omega))_t + (hu(v+\Omega))_x = 0$, with $\Omega_x = f$, which is strongly related to the potential vorticity :

$$F^{hv}_{i+1/2-} = \begin{cases} F^h_{i+1/2} v_i & \text{if } F^h_{i+1/2} \geq 0, \\ F^h_{i+1/2}(v_{i+1} + \Delta\Omega_{i+1/2}) & \text{if } F^h_{i+1/2} \leq 0, \end{cases} \tag{13}$$

$$F^{hv}_{i+1/2+} = \begin{cases} F^h_{i+1/2}(v_i - \Delta\Omega_{i+1/2}) & \text{if } F^h_{i+1/2} \geq 0, \\ F^h_{i+1/2} v_{i+1} & \text{if } F^h_{i+1/2} \leq 0, \end{cases} \tag{14}$$

with

$$\Delta\Omega_{i+1/2} = f_{i+1/2}\Delta x_{i+1/2}. \tag{15}$$

Second-order reconstruction - general

Reconstruction operator : $(U_i, Z_i) \to U_{i+1/2-}, Z_{i+1/2-}, U_{i+1/2+}, Z_{i+1/2+}$ for $i \in \mathbb{Z}$, in a way that it is

- conservative in U :

$$\frac{U_{i-1/2+} + U_{i+1/2-}}{2} = U_i, \tag{16}$$

- second-order, i.e. that whenever for all i,

$$U_i = \frac{1}{\Delta x_i} \int_{C_i} U(x)\,dx, \qquad Z_i = \frac{1}{\Delta x_i} \int_{C_i} Z(x)\,dx, \tag{17}$$

  for smooth $U(x)$, $Z(x)$, then, for $\delta = \sup_i \Delta x_i$

$$\begin{aligned} U_{i+1/2-} &= U(x_{i+1/2}) + O(\delta^2), \qquad & U_{i+1/2+} &= U(x_{i+1/2}) + O(\delta^2), \\ Z_{i+1/2-} &= Z(x_{i+1/2}) + O(\delta^2), \qquad & Z_{i+1/2+} &= Z(x_{i+1/2}) + O(\delta^2). \end{aligned} \tag{18}$$

Possible reconstructions : *minmod* (respects max principle), *ENO* (non-oscillatory), ...

Second-order scheme :

$$U_i^{n+1} - U_i^n + \frac{\Delta t}{\Delta x_i}(F_{i+1/2-} - F_{i-1/2+} - F_i) = 0, \qquad (19)$$

with

$$\begin{aligned} F_{i+1/2-} &= \mathcal{F}_l\left(U^n_{i+1/2-}, U^n_{i+1/2+}, Z^n_{i+1/2-}, Z^n_{i+1/2+}\right), \\ F_{i+1/2+} &= \mathcal{F}_r\left(U^n_{i+1/2-}, U^n_{i+1/2+}, Z^n_{i+1/2-}, Z^n_{i+1/2+}\right), \\ F_i &= \mathcal{F}_c\left(U^n_{i-1/2+}, U^n_{i+1/2-}, Z^n_{i-1/2+}, Z^n_{i+1/2-}\right), \end{aligned} \qquad (20)$$

where the arguments are obtained from the reconstruction operator applied to (U_i^n, Z_i^n), and the centered flux function $\mathcal{F}_c$ to be chosen.

2nd-order reconstruction for shallow water

$(U_i, z_i) = (h_i, h_i u_i, z_i)$
$U_{i+1/2\pm} \equiv (h_{i+1/2\pm}, h_{i+1/2\pm} u_{i+1/2\pm})$ -reconstructed values. Then

$$\frac{h_{i-1/2+} + h_{i+1/2-}}{2} = h_i, \qquad (21)$$
$$\frac{h_{i-1/2+} u_{i-1/2+} + h_{i+1/2-} u_{i+1/2-}}{2} = h_i u_i.$$

Equivalent to

$$h_{i-1/2+} = h_i - \frac{\Delta x_i}{2} Dh_i, \qquad h_{i+1/2-} = h_i + \frac{\Delta x_i}{2} Dh_i,$$
$$u_{i-1/2+} = u_i - \frac{h_{i+1/2-}}{h_i} \frac{\Delta x_i}{2} Du_i,$$
$$u_{i+1/2-} = u_i + \frac{h_{i-1/2+}}{h_i} \frac{\Delta x_i}{2} Du_i, \qquad (22)$$

for some slopes Dh_i, Du_i. *Minmod*, *ENO*, ENO_m for them. ENO_m for $h + z$ variable.
Centered flux : $\mathcal{F}_c(U_l, U_r, \Delta z) = \left(0, -\frac{h_l + h_r}{2} g \Delta z\right)$.

Second-order accuracy in time

The second-order accuracy in time can be obtained by the Heun method. The second-order scheme in x can be written as

$$U^{n+1} = U^n + \Delta t\, \Phi(U^n), \tag{23}$$

where $U = (U_i)_{i \in \mathbb{Z}}$, and Φ is a nonlinear operator depending on the mesh. Then the second-order scheme in time is

$$\begin{aligned} \widetilde{U}^{n+1} &= U^n + \Delta t\, \Phi(U^n), \\ \widetilde{U}^{n+2} &= \widetilde{U}^{n+1} + \Delta t\, \Phi(\widetilde{U}^{n+1}), \\ U^{n+1} &= \frac{U^n + \widetilde{U}^{n+2}}{2}. \end{aligned} \tag{24}$$

If the operator Φ does not depend on Δt, this procedure gives a fully second-order scheme in space and time. The convex average in (24) enables to ensure the stability without any further limitation on the CFL.

From 1- to 2-dimensions :

Our interest - systems of the form :

$$\partial_t U + \partial_x(F_1(U,Z)) + \partial_y(F_2(U,Z)) + B_1(U,Z)\partial_x Z + B_2(U,Z)\partial_y Z = 0. \tag{25}$$

2d quasilinear system :

$$\partial_t U + A_1 U \partial_x U + A_2 U \partial_y U = 0. \tag{26}$$

Consider planar solutions of the form $U(t,x,y) = U(t,\zeta)$ with $\zeta = xn^1 + yn^2$ and (n^1, n^2) is a unit vector, which leads to

$$\partial_t U + A_n(U)\partial_\zeta U = 0, \tag{27}$$

with

$$A_n U = n^1 A_1(U) + n^2 A_2(U). \tag{28}$$

The notions introduced for one-dimensional systems can be applied to (27), and one defines hyperbolicity, entropies, and other notions for (26) by defining them for all directions n.

2-dimensional mesh :

Rectangles

$$C_{ij} = (x_{i-1/2}, x_{i+1/2}) \times (y_{j-1/2}, y_{j+1/2}), \qquad i \in \mathbb{Z},\ j \in \mathbb{Z}, \tag{29}$$

with sides :

$$\Delta x_i = x_{i+1/2} - x_{i-1/2} > 0, \qquad \Delta y_j = y_{j+1/2} - y_{j-1/2} > 0. \tag{30}$$

The centers of the cells : $x_{ij} = (x_i, y_j)$, with

$$x_i = \frac{x_{i-1/2} + x_{i+1/2}}{2}, \qquad y_j = \frac{y_{j-1/2} + y_{j+1/2}}{2}. \tag{31}$$

Finite-volumes in 2 dimensions

Goal : to approximate solution $U(t,x,y)$ to (25) by discrete values U_{ij}^n that are approximations of the mean value of U over the cell C_{ij} at time $t_n = n\Delta t$,

$$U_{ij}^n \simeq \frac{1}{\Delta x_i \Delta y_j} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} U(t_n,x,y)\, dxdy. \qquad (32)$$

A finite volume method for solving (25) takes the form

$$U_{ij}^{n+1} - U_{ij}^n + \frac{\Delta t}{\Delta x_i}(F_{i+1/2-,j} - F_{i-1/2+,j}) + \frac{\Delta t}{\Delta y_j}(F_{i,j+1/2-} - F_{i,j-1/2+}) = 0, \qquad (33)$$

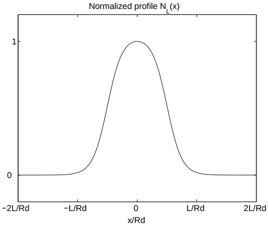
Exchange terms :

$$\begin{aligned} F_{i+1/2\mp,j} &= \mathcal{F}^1_{l/r}(U_{ij}, U_{i+1,j}, Z_{ij}, Z_{i+1,j}), \\ F_{i,j+1/2\mp} &= \mathcal{F}^2_{l/r}(U_{ij}, U_{i,j+1}, Z_{ij}, Z_{i,j+1}), \end{aligned} \qquad (34)$$

for some numerical fluxes $\mathcal{F}^1_l$, $\mathcal{F}^1_r$, $\mathcal{F}^2_l$, $\mathcal{F}^2_r$.

Jet profile

A strongly unbalanced jet (no initial pressure perturbation), $Ro \sim 1$

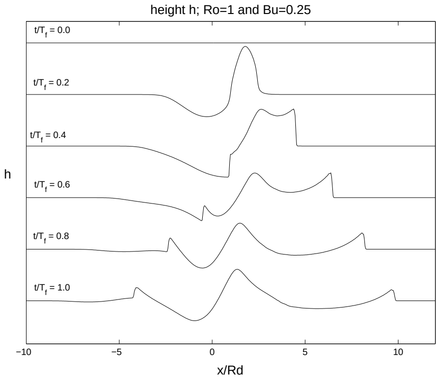


Crucial parameters : Rossby and Burger numbers :

$$Ro = \frac{V}{fL} \quad , \quad Bu = \frac{gH}{f^2 L^2}. \tag{35}$$

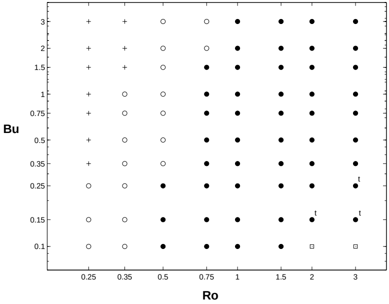
Adjusts by emitting inertia-gravity waves forming shocks. The PV - bearing part of the flow should reache an equilibrium state.

Snapshots of the jet adjustment



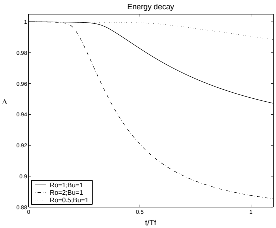
Two shocks are formed at $t = 0.3$ and propagate to the left and to the right from the jet,respectively. One of the shocks is formed immediately within the jet core.

Statistics of the shock formation



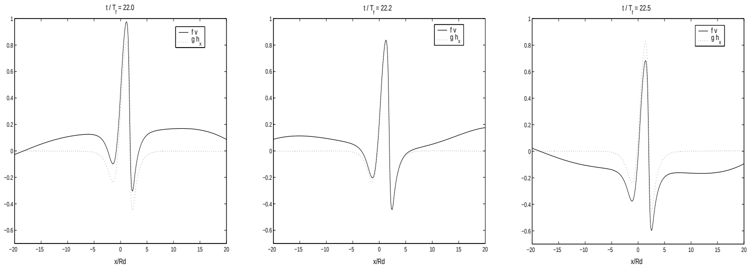
Breaking in $t < \frac{\pi}{f}$: *shaded circles*; breaking in $\frac{\pi}{f} < t < \frac{2\pi}{f}$: *open circles*; breaking in $t > \frac{2\pi}{f}$: *crosses*. Appearance of transonic shocks with propagation velocity changing sign in course of evolution : *superscript t*. Drying was observed for large *Ro* and small *Bu* : *squares*.

Energy conservation/dissipation



Shock-induced energy decay in jet adjustment. Evolution of the nondimensional energy anomaly $\Delta = (e - e_p(0))/e_p(0)$ with $e_p(t) = \frac{1}{2}\int dxg(h-H)^2 d$ and $e = e_p + \frac{1}{2}\int dx\, h(u^2+v^2)$ computed in the volume $[-5L, 5L]$

Check of balance



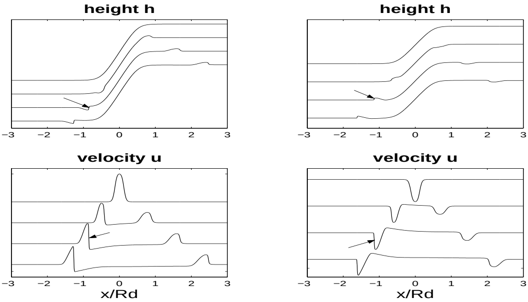
Mean state in geostrophic balance (middle panel) is rapidly achieved, small oscillations persist in the jet core, with amplitude decreasing with time and depending on *Ro* and *Bu*. The period of oscillations is close to $T_f \Rightarrow$ inertial oscillations.

Check of the breaking theory

Riemann invariants in Eulerian variables :

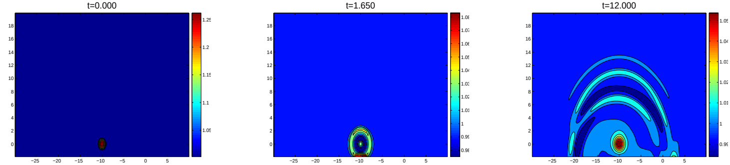
$$R_{\pm} = 8\left(\frac{H}{h}\right)^{1/4} \partial_x u \pm \sqrt{\frac{g}{H}}\left(\frac{h}{H}\right)^{3/4} \partial_x h$$

- dominated by the 2nd term. Part of the perturbation going towards small h breaks first.



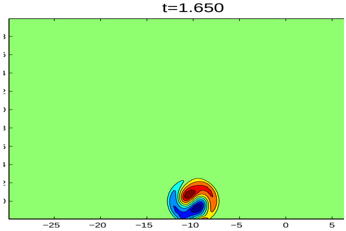
Wave-breaking in a balanced jet with $Ro = Bu = 1$. Perturbation in u with $Ro_p = 0.8$.

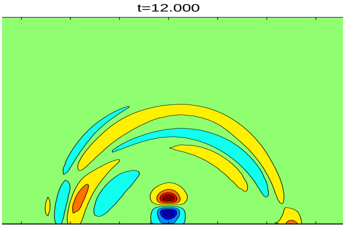
Classical example : adjustment of a bump in *h* (near a border) ; height field



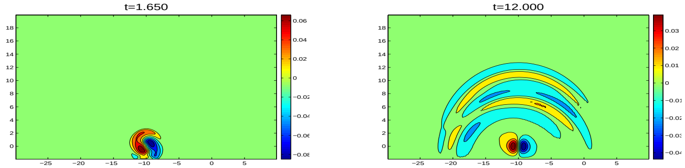
Inertia-gravity wave emission + Kelvin wave.

Adjustment of a bump in h near border ; u - field

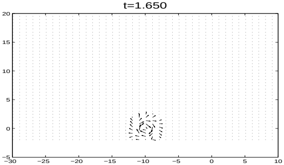


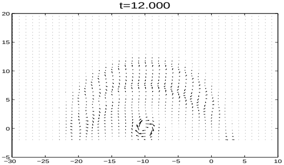


Adjustment of a bump in h near border ; v - field

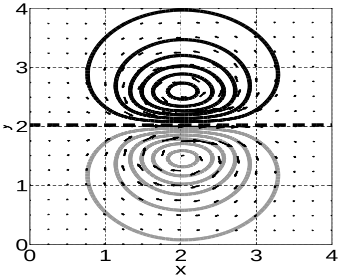


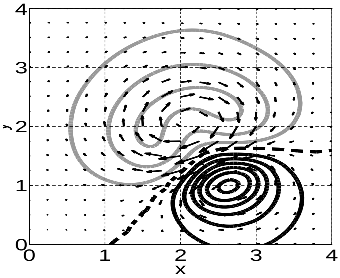
Adjustment of a bump in *h* near border ; check of balance





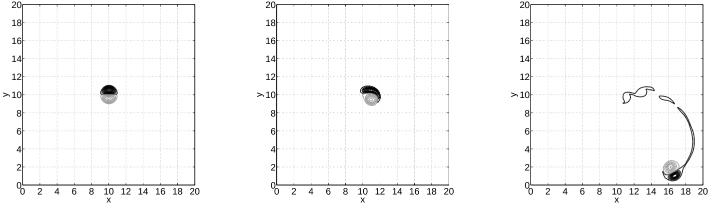
From balanced to unbalanced dipole





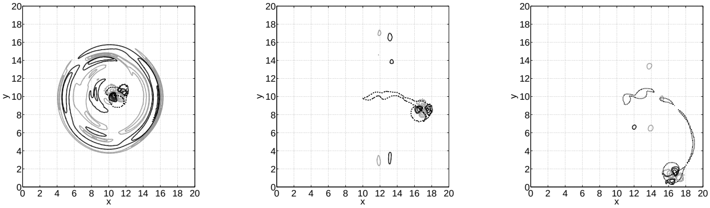
Initial balanced (*left*) and late (t = 1OO/f) (*right*) configurations.

Ageostrophic adjustment seen in PV field



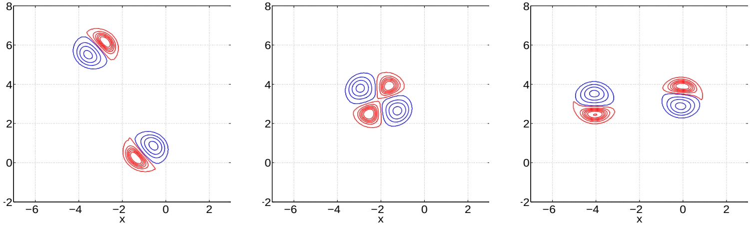
Adjustment of a balanced dipole at $Ro = 0.2$: PV at $t = 0, 10, 100$, from left to right. Black : cyclone, gray : anticyclone. PV anomaly taking values in the interval $[-2.7; 5.1]$

Adjustment as seen in the divergence field.



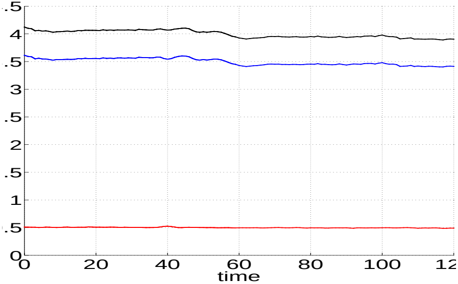
$t = 5, 45, 100$, from left to right. Contours of PV anomaly $|Q| = 0.05$ superimposed. Black : divergence, gray : convergence. Divergence taking values in the interval $[-0.2; 0.3]$

Collisions of ageostrophic dipoles



Evolution of PV during the collision : $t = 25, 40, 50$, from left to right.

Energy evolution during collision



Total (black), kinetic (blue) and potential (red) energy during the collision.

Multi-layer 1-dimensional shallow-water system with topography in conservative form

$$\partial_t h_j + \partial_x \left(h_j u_j \right) = 0, \tag{36}$$

$$\partial_t \left(h_j u_j \right) + \partial_x \left(h_j u_j^2 + g h_j^2/2 \right) + g h_j \left(z + \sum_{k>j} h_k + \sum_{k<j} \frac{\rho_k}{\rho_j} h_k \right) = 0, \tag{37}$$

where $h_j \geq 0$, $j = 1, 3, ...m$ - layer depths, u_j - layer velocities, $z(x)$ - topography, and

$$0 < \rho_1 \leq ... \leq \rho_m$$

layer densities. Convex entropy $\equiv$ energy.

Equivalent 1-layer systems

Shallow-water systems for $U^j = (h_j, h_j u_j)$ with effective topography :

$$z^j = z + \sum_{k>j} h_k + \sum_{k<j} \frac{\rho_k}{\rho_j} h_k$$

Finite volume scheme with numerical fluxes $\mathcal{F}_{l/r}$:

$$U_i^{j,n+1} - U_i^j + \frac{\Delta t}{\Delta x_i} \left(\mathcal{F}_l(U_i^j, U_{i+1}^j, z_i^j, z_{i+1}^j) - \mathcal{F}_r(U_{i-1}^j, U_i^j, z_{i-1}^j, z_i^j) \right) = 0. \quad (38)$$

For each j - effective shallow water $\Rightarrow$ well-balancing, hydrostatic reconstruction, apparent topography etc will be applied.

Splitting vs sum methods, an example

Consider an ODE

$$\frac{dU}{dt} + A(U) + B(U) = 0. \tag{39}$$

Solving $dU/dt + A(U) = 0$, and $dU/dt + B(U) = 0$, resp. :

$$\begin{aligned} U^{n+1} - U^n + \Delta t\, A(U^n) &= 0, \\ U^{n+1} - U^n + \Delta t\, B(U^n) &= 0. \end{aligned}$$

Splitting method :

$$\begin{aligned} U^{n+1/2} - U^n + \Delta t\, A(U^n) &= 0, \\ U^{n+1} - U^{n+1/2} + \Delta t\, B(U^{n+1/2}) &= 0. \end{aligned}$$

- solving (38) successively.

Sum method

$$U^{n+1} - U^n + \Delta t\ (A(U^n) + B(U^n)) = 0,$$

- solving (38) simultaneously

References, finite-volume wall-balanced numerical schemes

1-layer

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References : tests

1.5 RSW

V. Zeitlin, S. Medvedev & R. Plougonven, 2003, "Frontal geostrophic adjustment, slow manifold and nonlinear wave phenomena in one-dimensional rotating shallow water. Part 1. Theory", *J. Fluid Mech.*, **481**, pp 35-63.

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