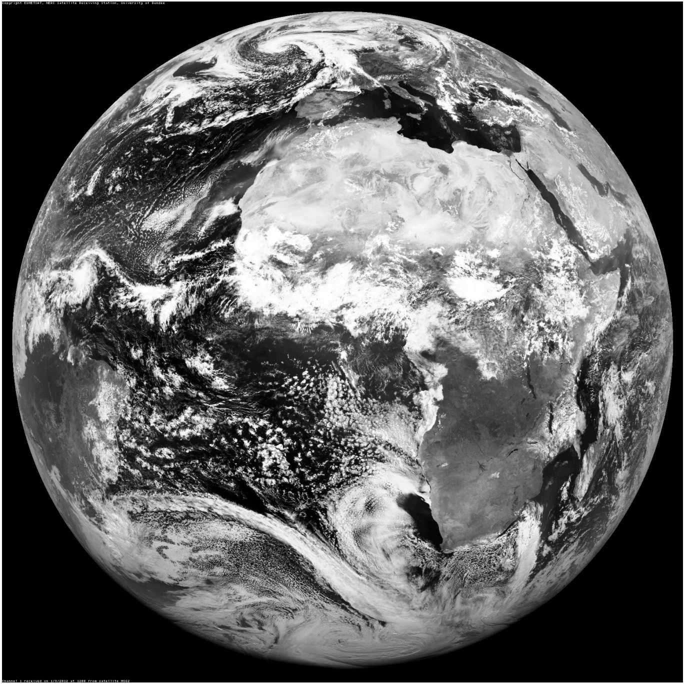
Understanding large-scale atmospheric and oceanic flows with layered rotating shallow water models

V. Zeitlin

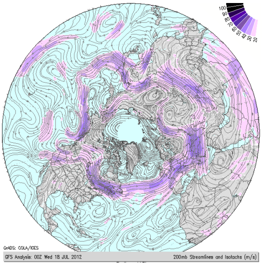
Laboratoire de Météorologie Dynamique, Paris

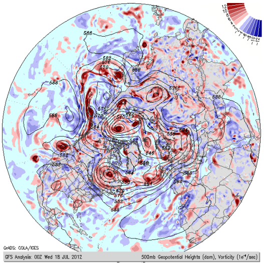
Non-homogeneous Fluids and Flows, Prague, August 2012

Plan


Copyright EUMETSAT, NERC Satellite Receiving Station, University of Dundee
Channel 1 received on 1/8/2012 at 1200 from satellite MSG2

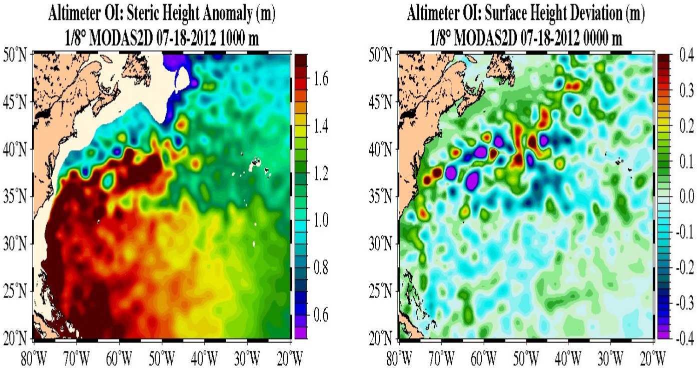
Midlatitude atmospheric jet





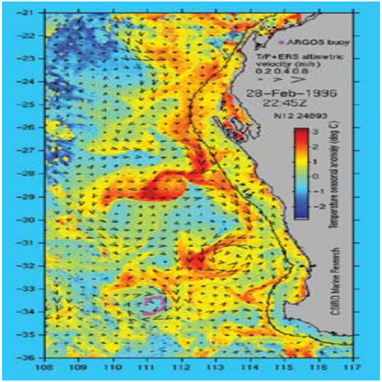
Midlatidude upper-tropospheric jet (left) and related synoptic systems (right).

Oceanic currents : Gulfstream



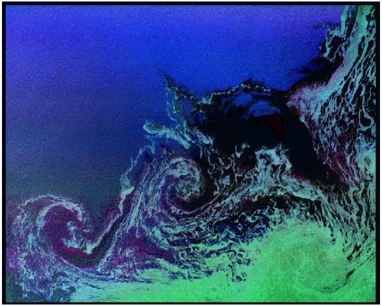
Gulfstream (left) and related vortices (right). Velocity follows isopleths of the height anomaly in the first approximation.

Coastal current and associated vortices



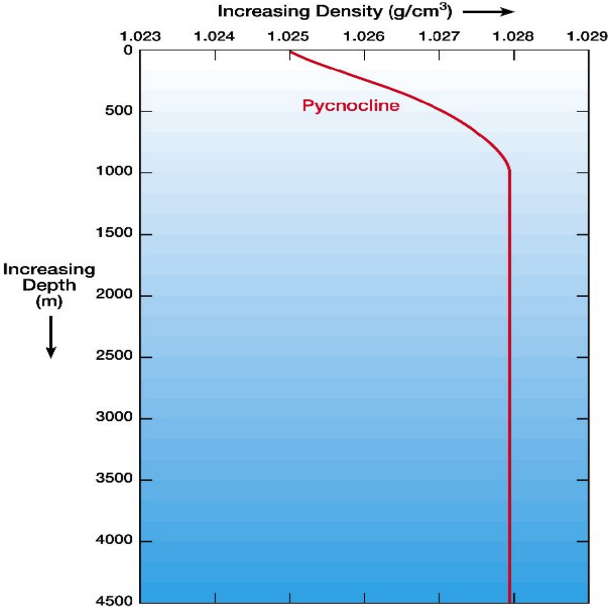
Velocity (arrows) and temperature anomaly (colors) of the Leeuwin curent near Australian coast.

Destabilizing coastal flow



Instability of a coastal current in the Weddell sea.

Mean oceanic stratification



Primitive equations : ocean

Hydrostatics

$$g\rho + \partial_z P = 0, \tag{1}$$

$$\begin{aligned} P &= P_0 + P_s(z) + \pi(x, y, z; t), \\ \rho &= \rho_0 + \rho_s(z) + \sigma(x, y, z; t), \quad \rho_0 \gg \rho_s \gg \sigma \end{aligned}$$

Incompressibility

$$\vec{\nabla} \cdot \vec{v} = 0, \quad \vec{v} = \vec{v}_h + \hat{z} w. \tag{2}$$

Euler :

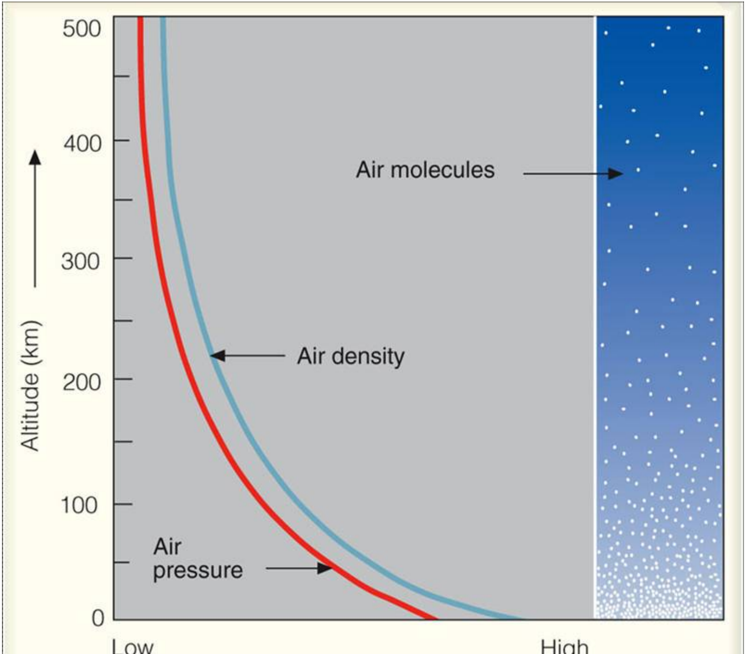
$$\frac{\partial \vec{v}_h}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}_h + f \hat{z} \wedge \vec{v}_h = -\vec{\nabla}_h \phi. \tag{3}$$

$\phi = \frac{\pi}{\rho_0}$ - geopotential.

Continuity :

$$\partial_t \rho + \vec{v} \cdot \vec{\nabla} \rho = 0. \tag{4}$$

Mean atmospheric stratification



Primitive equations : atmosphere, pseudo-height vertical coordinate

$$\frac{\partial \vec{v}_h}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}_h + f \hat{z} \wedge \vec{v}_h = -\vec{\nabla}_h \phi, \tag{5}$$

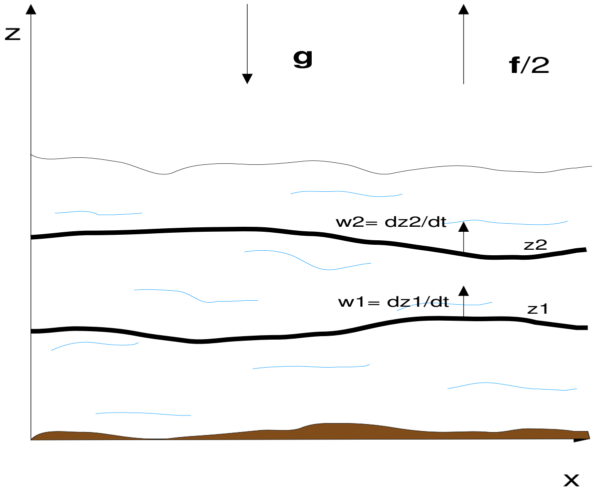
$$-g \frac{\theta}{\theta_0} + \frac{\partial \phi}{\partial z} = 0, \tag{6}$$

$$\frac{\partial \theta}{\partial t} + \vec{v} \cdot \vec{\nabla} \theta = 0; \quad \vec{\nabla} \cdot \vec{v} = 0. \tag{7}$$

Identical to oceanic ones with $\sigma \to -\theta$, potential temperature.
Vertical coordinate : pseudo-height, P - pressure.

$$\bar{z} = z_0 \left(1 - \left(\frac{P}{P_s} \right)^{\frac{R}{c_p}} \right) \tag{8}$$

Material surfaces



Vertical averaging and RSW models

- Take horizontal momentum equation in conservative form :

$$(\rho u)_t + (\rho u^2)_x + (\rho v u)_y + (\rho w u)_z - f\rho v = -p_x, \quad (9)$$

and integrate between a pair of material surfaces $z_{1,2}$:

$$w|_{z_i} = \frac{dz_i}{dt} = \partial_t z_i + u\partial_x z_i + v\partial_y z_i, \quad i = 1, 2. \quad (10)$$

- Use Leibnitz formula and get :

$$\partial_t \int_{z_1}^{z_2} dz \rho u \; + \; \partial_x \int_{z_1}^{z_2} dz \rho u^2 + \partial_y \int_{z_1}^{z_2} dz \rho u v -$$
$$f \int_{z_1}^{z_2} dz \rho v \; = \; -\partial_x \int_{z_1}^{z_2} dz p - \partial_x z_1 \; p|_{z_1} + \partial_x z_2 \; p|_{z_2} \, .$$

(analogously for v).

- Use continuity equation and get

$$\partial_t \int_{z_1}^{z_2} dz\rho + \partial_x \int_{z_1}^{z_2} dz\rho u + \partial_y \int_{z_1}^{z_2} dz\rho v = 0. \quad (11)$$

- Introduce the mass- (entropy)- averages :

$$\langle F \rangle = \frac{1}{\mu} \int_{z_1}^{z_2} dz\rho F, \quad \mu = \int_{z_1}^{z_2} dz\rho. \quad (12)$$

and obtain averaged equations :

$$\begin{aligned} & \partial_t (\mu\langle u \rangle) + \partial_x \left(\mu\langle u^2 \rangle\right) + \partial_y (\mu\langle uv \rangle) - f\mu\langle v \rangle \\ = \ & -\partial_x \int_{z_1}^{z_2} dzp - \partial_x z_1 \, p|_{z_1} + \partial_x z_2 \, p|_{z_2} , \end{aligned} \quad (13)$$

$$\begin{aligned} & \partial_t (\mu\langle v \rangle) + \partial_x (\mu\langle uv \rangle) + \partial_y \left(\mu\langle v^2 \rangle\right) + f\mu\langle u \rangle \\ = \ & -\partial_y \int_{z_1}^{z_2} dzp - \partial_y z_1 \, p|_{z_1} + \partial_y z_2 \, p|_{z_2} , \end{aligned} \quad (14)$$

$$\partial_t \mu + \partial_x (\mu\langle u \rangle) + \partial_y (\mu\langle v \rangle) = 0. \quad (15)$$

- Use hydrostatics and get, introducing mean constant density $\bar{\rho}$:

$$p(x,y,z,t) \approx -g\bar{\rho}(z - z_1) + p|_{z_1} . \qquad (16)$$

- Use the mean-field (= columnar motion) approximation :

$$\langle uv\rangle \approx \langle u\rangle\langle v\rangle,\ \langle u^2\rangle \approx \langle u\rangle\langle u\rangle,\ \langle v^2\rangle \approx \langle v\rangle\langle v\rangle. \qquad (17)$$

and get master equation for the layer :

$$\begin{aligned} & \bar{\rho}(z_2 - z_1)(\partial_t \mathbf{v}_h + \mathbf{v}\cdot\nabla\mathbf{v}_h + f\hat{\mathbf{z}}\wedge\mathbf{v}_h) = \\ - \quad & \nabla_h\left(-g\bar{\rho}\frac{(z_2 - z_1)^2}{2} + (z_2 - z_1)\, p|_{z_1}\right) \\ - \quad & \nabla_h z_1\, p|_{z_1} + \nabla_h z_2\, p|_{z_2} . \end{aligned} \qquad (18)$$

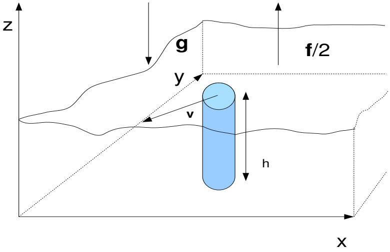
- Pile up layers, with lowermost boundary fixed by topography, and uppermost free or fixed.

1-layer RSW, $z_1 = 0$, $z_2 = h$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + f\hat{\mathbf{z}} \wedge \mathbf{v} + g \nabla h = 0\,, \tag{19}$$

$$\partial_t h + \nabla \cdot (\mathbf{v} h) = 0\,. \tag{20}$$

$\Rightarrow$ 2d barotropic gas dynamics + Coriolis force.
In the presence of nontrivial topography $b(x, y)$:
$h \to h - b$ in the second equation.



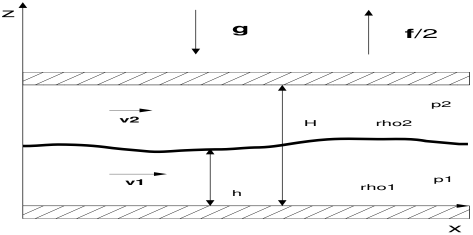
2-layer RSW, rigid lid : $z_1 = 0$, $z_2 = h$, $z_3 = H = \text{const}$

$$\partial_t \mathbf{v}_i + \mathbf{v}_i \cdot \nabla \mathbf{v}_i + f\hat{\mathbf{z}} \wedge \mathbf{v}_i + \frac{1}{\bar{\rho}_i}\nabla \pi_i = 0\,, i = 1, 2; \tag{21}$$

$$\partial_t h + \nabla \cdot (\mathbf{v}_1 h) = 0\,, \tag{22}$$

$$\partial_t (H - h) + \nabla \cdot (\mathbf{v}_2 (H - h)) = 0\,, \tag{23}$$

$$\pi_1 = (\bar{\rho}_1 - \bar{\rho}_2) g h + \pi_2\,. \tag{24}$$



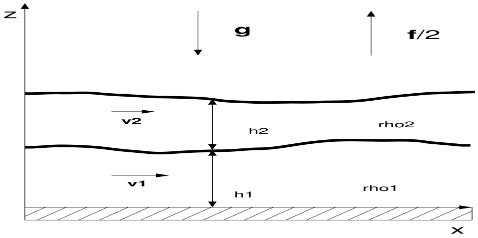
2-layer rotating shallow water model with a free surface : $z_1 = 0$, $z_2 = h_1$, $z_3 = h_1 + h_2$

$$\partial_t \mathbf{v}_2 + \mathbf{v}_2 \cdot \nabla \mathbf{v}_2 + f\hat{\mathbf{z}} \wedge \mathbf{v}_2 = -\nabla(h_1 + h_2) \tag{25}$$

$$\partial_t \mathbf{v}_1 + \mathbf{v}_1 \cdot \nabla \mathbf{v}_1 + f\hat{\mathbf{z}} \wedge \mathbf{v}_1 = -\nabla(rh_1 + h_2), \tag{26}$$

$$\partial_t h_{1,2} + \nabla \cdot \left(\mathbf{v}_{1,2} h_{1,2}\right) = 0\,, \tag{27}$$

where $r = \frac{\rho_1}{\rho_2} \leq 1$ - density ratio, and $h_{1,2}$ - thicknesses of the layers.



Useful notions

Balanced vs unbalanced motions

Geostrophic balance : balance between the Coriolis force and the pressure force. In shallow-water model :

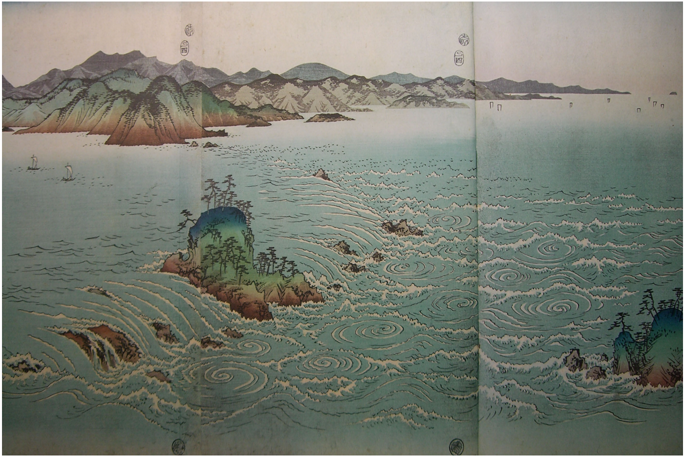
$$f\hat{\mathbf{z}} = -g\nabla h \tag{28}$$

Valid at small Rossby numbers : $Ro = U/fL$, where U, L - characteristic velocity and horizontal scale. Balanced motions at small Ro : vortices. Unbalanced motions : inertia-gravity waves.

Relative, absolute and potential vorticity

Relative vorticity in layered models : $\zeta = \hat{\mathbf{z}} \cdot \nabla \wedge \mathbf{v}$.
Absolute vorticity : $\zeta + f$. Potential vorticity (PV) : $q_{12} = \frac{\zeta+f}{z_2-z_1}$ for the fluid layer between z_2 and z_1.

Dynamical actors in RSW : vortices & waves



Conservation laws (no topography)

Equations in conservative form (momentum & mass)

$$
\begin{aligned}
\partial_t(hu) + \partial_x(hu^2) + \partial_y(huv) - fhv + g\partial_x \frac{h^2}{2} &= 0,\\
\partial_t(hv) + \partial_x(huv) + \partial_y(hv^2) + fhu + g\partial_y \frac{h^2}{2} &= 0,\\
\partial_t h + \partial_x(hu) + \partial_y(hv) &= 0.
\end{aligned}
$$

Energy

$$
E = \int dxdy\, e = \int dxdy\, \left(h\frac{u^2+v^2}{2} + g\frac{h^2}{2} \right),
$$

is locally conserved : $\partial_t e + \nabla \cdot \mathbf{f_e} = 0$.

Lagrangian conservation of potential vorticity : $(\partial_t + u\partial_x + v\partial_y)\, q = 0$.

Lagrangian view of RSW dynamics

Mapping : $(x, y) \longrightarrow (X(x, y; t), Y(x, y; t))$
Equations of motion (dot - Lagrangian derivative) :

$$\begin{aligned} \ddot{X} - f\dot{Y} &= -g\partial_X h, \\ \ddot{Y} + f\dot{X} &= -g\partial_Y h. \end{aligned} \tag{29}$$

Mass conservation : $h_I(x, y) = h(X, Y)\mathcal{J}(X, Y)$, where $h_I = h(x, y; 0)$, $\mathcal{J}(X, Y) = \frac{\partial(X,Y)}{\partial(x,y)}$ - Jacobian of the mapping.
Hence :

$$\partial_X h = \frac{\partial(h, Y)}{\partial(X, Y)} = \frac{\partial(h, Y)}{\partial(x, y)} \cdot (\mathcal{J}(X, Y))^{-1} = \mathcal{J}\left(h_I, (\mathcal{J}(X, Y))^{-1}\right), \tag{30}$$

ant similarly for $\partial_Y h$.

"1.5 dimensional" RSW

$$\begin{aligned} \ddot{X} - fv + g\frac{\partial h}{\partial X} &= 0\,, \\ (v + \dot{fX}) &= 0\,, \end{aligned} \tag{31}$$

with $h(X,t) = h_I(x)\frac{\partial x}{\partial X}$. Hence, v is not an independent variable.

Alternatively ($h_I = H =$ const, for simplicity ; h_I may be always "straightened" by additional change of variables $x \to a$) :

$$\dot{u} - fv + gH\frac{\partial}{\partial a}\frac{1}{2J^2} = 0,$$

$$\dot{v} + fu = 0,$$

$$\dot{J} - \frac{\partial u}{\partial a} = 0.$$

Here $J = \frac{\partial X}{\partial a} = \frac{H}{h(X,t)}$, $P = \frac{gH}{2J^2}$ - Lagrangian pressure.

Reduction

Rewrite the Lagrangian equations of motion as a system of 2 equations

$$\partial_t u + \partial_a P = v\,, \qquad (32)$$
$$\partial_t J - \partial_a u = 0\,, \qquad (33)$$

where v is not an independent variable and is given by $\partial_a v = Q(a) - J$, $Q(a)$ - potential vorticity : $Q(a) = \frac{1}{H}\left(\frac{\partial v}{\partial a} + fJ\right) = \frac{1}{H}\left(\frac{\partial v_I}{\partial a} + fJ_I\right)$. and the Lagrangian time-derivative is denoted by ∂_t. (Dimensionful parameters will be omitted ; correct dimensions easy to recover.)

Quasilinear system

$$\begin{pmatrix} u \\ J \end{pmatrix}_t + \begin{pmatrix} 0 & -J^{-3} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ J \end{pmatrix}_a = \begin{pmatrix} v \\ 0 \end{pmatrix} . \qquad (34)$$

Eigenvalues and left eigenvectors : $\mu_\pm = \pm J^{-\frac{3}{2}}$ and $\left(1\ , \pm J^{-\frac{3}{2}}\right)$. Riemann invariants : $w_\pm = u \pm 2J^{-\frac{1}{2}}$,

$$\partial_t w_\pm + \mu_\pm \partial_a w_\pm = v\,. \qquad (35)$$

Original variables in terms of $w_\pm$:

$$u = \frac{1}{2}(w_+ + w_-)\,, \qquad (36)$$

$$J = \frac{16}{(w_+ - w_-)^2} > 0\,, \qquad (37)$$

$$\mu_\pm = \pm\left(\frac{w_+ - w_-}{4}\right)^3 . \qquad (38)$$

In terms of the derivatives of the Riemann invariants $r_\pm = \partial_a w_\pm$ we get

$$\partial_t r_\pm + \mu_\pm \partial_a r_\pm + \frac{\partial \mu_\pm}{\partial w_+} r_+ r_\pm + \frac{\partial \mu_\pm}{\partial w_-} r_- r_\pm = \partial_a v = Q(a) - J\,, \tag{39}$$

which may be rewritten using "double Lagrangian" derivatives $\frac{d}{dt_\pm} = \partial_t + \mu_\pm \partial_a$ as

$$\frac{dr_\pm}{dt_\pm} + \frac{\partial \mu_\pm}{\partial w_+} r_+ r_\pm + \frac{\partial \mu_\pm}{\partial w_-} r_- r_\pm = Q(a) - J\,, \tag{40}$$

and breaking and shock formation correspond to in $r_\pm \to \pm\infty$ in finite time.

Ricatti equation

In terms of new variables $R_\pm = e^\lambda r_\pm$, with $\lambda = \frac{3}{128}\log|w_+ - w_-|$, equations (39) may be rewritten as

$$\frac{dR_\pm}{dt_\pm} = -e^{-\lambda}\frac{\partial \mu_\pm}{\partial w_\pm}R_\pm^2 + e^\lambda\left(Q(a) - J\right), \qquad (41)$$

where $\frac{\partial \mu_\pm}{\partial w_\pm} = \frac{3}{64}(w_+ - w_-)^2 > 0$.
This is a generalized Ricatti equation and from its qualitative analysis it follows that :

1. if initial *relative vorticity* $Q - J = \partial_a v$ is sufficiently **negative**, **breaking** takes place whatever initial conditions are
2. if the *relative vorticity* is **positive** as well as the derivatives of the Riemann invariants at the initial moment, there is **no breaking**.

Conservation laws of the 2-layer RSW (no topography)

- Overall momentum (modulo Coriolis force)
- Mass layerwise
- Energy :

$$
\begin{aligned}
E &= \int dxdy \Big[\left(\frac{\rho_1}{2} h_1 \mathbf{v_1}^2 + \frac{\rho_2}{2} h_2 \mathbf{v_2}^2 \right) \\
&+ \left(\frac{\rho_1}{2} g h_1^2 + \frac{\rho_2}{2} g h_2^2 \right) + \rho_1 g h_1 h_2 \Big]
\end{aligned}
$$

- Potential vorticity layerwise, Lagrangian :

$$
q_1 = \frac{f + \partial_x v_1 - \partial_y u_1}{h_1} \quad , \quad q_2 = \frac{f + \partial_x v_2 - \partial_y u_2}{h_2},
$$

2-layer 1.5d RSW model with a rigid lid

Basic equations :

$$
\begin{aligned}
\partial_t u_1 + u_1 \partial_x u_1 - f v_1 + \rho_1^{-1} \partial_x \pi_1 &= 0 \, , \\
\partial_t v_1 + u_1 (f + \partial_x v_1) &= 0 \, , \\
\partial_t u_2 + u_2 \partial_x u_2 - f v_2 + \rho_2^{-1} \partial_x \pi_1 + g' \partial_x \eta &= 0 \, , \\
\partial_t v_2 + u_2 (f + \partial_x v_2) &= 0 \, , \\
\partial_t (H_1 - \eta) + \partial_x ((H_1 - \eta) u_1) &= 0 \, , \\
\partial_t (H_2 + \eta) + \partial_x ((H_2 + \eta) u_2) &= 0 \, ,
\end{aligned}
$$

$(u_1, \ v_1)$, $(u_2, \ v_2)$ velocities in upper/lower layer ; $\pi_2 = \pi_1 + g(\rho_1 h_1 + \rho_2 h_2)$, η - interface displacement, H_1 and H_2 - heights of two layers at rest ; possible to use the full layers' heights $h_{1,2} = H_{1,2} \mp \eta$ as dynamical variables, g' is reduced gravity : $g' = g(\rho_2 - \rho_1)/\rho_2$.

4 equations for 4 variables $u_2,\ h_2,\ v_2,\ v_1$

Eliminating barotropic pressure and imposing zero transverse momentum :

$$\frac{du_2}{dt} - fv_2 \quad + \quad \frac{\rho_1}{\rho_2 h_1 + \rho_1 h_2}\Bigg(f(h_1 v_1 + h_2 v_2) \\ - \quad \partial_x \left(h_1 u_1^2 + h_2 u_2^2 \right) + \frac{g\Delta\rho}{\rho_1} h_1 \partial_x h_2 \Bigg) = 0,$$

$$\frac{dh_2}{dt} + h_2 \partial_x u_2 = 0, \quad \frac{dv_2}{dt} + fu_2 = 0,$$

$$\frac{dv_1}{dt} + (u_1 - u_2)\partial_x v_1 + fu_1 = 0,$$

$$u_1 = \frac{h_2 u_2}{h_2 - H}, \quad h_1 = H - h_2.$$

Here $\frac{d}{dt} = \partial_t + u_2 \partial_x$ -Lagrangian derivative.

Further simplifications

Use of the mass Lagrangian variable a : $h_2 = \partial a/\partial x$) :

$$\frac{du_2}{dt} - fv_2 \quad + \quad \frac{\rho_1}{\rho_2 h_1 + \rho_1 h_2}\Bigg(f(h_1 v_1 + h_2 v_2)$$

$$- \quad h_2 \frac{\partial}{\partial a}\left(h_1 u_1^2 + h_2 u_2^2\right) + \frac{g\Delta\rho}{\rho_1} h_1 h_2 \partial_a h_2 \Bigg) = 0,$$

$$\frac{dh_2}{dt} + h_2 \partial_a u_2 = 0,$$

$$\frac{dv_2}{dt} + fu_2 = 0,$$

$$\frac{dv_1}{dt} + (u_1 - u_2) h_2 \partial_a v_1 + fu_1 = 0.$$

Matrix form

$$\frac{d}{dt}\begin{pmatrix} v_2 \\ v_1 \\ h_2 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & T & 0 & 0 \\ 0 & 0 & 0 & h_2^2 \\ 0 & 0 & M & 2N \end{pmatrix} \frac{\partial}{\partial a}\begin{pmatrix} v_2 \\ v_1 \\ h_2 \\ u_2 \end{pmatrix} = \begin{pmatrix} -fu_2 \\ -fu_1 \\ 0 \\ fh_1 \frac{\rho_2 v_2 - \rho_1 v_1}{\rho_2 h_1 + \rho_1 h_2} \end{pmatrix},$$

$$T = (u_1 - u_2)h_2 = \frac{Hu_2}{H - h_2} h_2,$$

$$M = \frac{\rho_1 h_2}{\rho_2 h_1 + \rho_1 h_2}\left(g\frac{\Delta\rho}{\rho_1}(H - h_2) - \frac{H^2 u_2^2}{(H - h_2)^2} \right),$$

$$N = -\frac{\rho_1 h_2}{\rho_2 h_1 + \rho_1 h_2} \frac{Hh_2}{H - h_2} u_2.$$

Eigenvalues and eigenvectors

Nontrivial eigenvalues (trivial ones : 0 and T) :

$$\det \begin{pmatrix} -\lambda & h_2^2 \\ M & 2N - \lambda \end{pmatrix} = \lambda^2 - 2N\lambda - h_2^2 M = 0 \tag{43}$$

and the solution

$$\lambda_\pm = N \pm \sqrt{N^2 + M h_2^2}. \tag{44}$$

The discriminant of this equation is

$$D = \frac{\rho_1 \rho_2 h_1 h_2^3}{(\rho_2 h_1 + \rho_1 h_2)^2} \left\{ g\Delta\rho \left(\frac{h_1}{\rho_1} + \frac{h_2}{\rho_2} \right) - \frac{H^2 u_2^2}{h_1^2} \right\}. \tag{45}$$

The eigenvalues (43) are real and, hence, the system is hyperbolic when D is positive.

Loss of hyperbolicity

If

$$(u_2 - u_1)^2 = \frac{H^2 u_2^2}{h_1^2} > g\Delta\rho \left(\frac{h_1}{\rho_1} + \frac{h_2}{\rho_2}\right) \qquad (46)$$

then $D < 0$ and the sytem loses hyperbolicity. The former condition may be rewritten as a condition on u_2

$$u_2^2 > g\Delta\rho \left(\frac{h_1}{\rho_1} + \frac{h_2}{\rho_2}\right) \frac{h_1^2}{H^2}.$$

We, thus, see that, unlike its one-layer counterpart, the 2-layer RSW changes type if the vertical shear of the transverse velocity is too strong. One may recognise in (46) the condition for Kelvin-Helmholtz (KH) instability. The KH instability is known to produce breaking of the growing interface wave with production of KH billows. This expected singularity is of different nature with respect to shock formation.

Literature

V. Zeitlin, ed. *"Nonlinear dynamics of rotating shallow water : methods and advances"*, Springer, NY, 2007, 391p.