

# **Introduction to numerical methods for hyperbolic conservation laws: FORCE-type schemes**

Eleuterio Toro  
Laboratory of Applied Mathematics  
University of Trento, Italy

[www.ing.unitn.it/toro](http://www.ing.unitn.it/toro)  
toro@ing.unitn.it

## The big picture: numerical methods to solve

$$\partial_t Q + \partial_x F(Q) + \partial_y G(Q) + \partial_z H(Q) = S(Q) + D(Q)$$

$$\partial_t Q + A(Q)\partial_x Q + B(Q)\partial_y Q + C(Q)\partial_z Q = S(Q) + D(Q)$$

Source terms  $S(Q)$  may be stiff

Advection terms may not admit a conservative form  
(nonconservative products)

Meshes are assumed unstructured

Very high order of accuracy in both space and time

May use upwind or centred approaches for numerical fluxes

Recall the integral form of the conservation laws

$$\partial_t Q + \partial_x F(Q) = 0$$

in a control volume  $[x_L, x_R] \times [t_1, t_2]$  is

$$\int_{x_L}^{x_R} Q(x, t_2) dx = \int_{x_L}^{x_R} Q(x, t_1) dx - \left[ \int_{t_1}^{t_2} F(Q(x_R, t)) dt - \int_{t_1}^{t_2} F(Q(x_L, t)) dt \right]$$

$$\frac{1}{\Delta x} \int_{x_L}^{x_R} Q(x, t_2) dx = \frac{1}{\Delta x} \int_{x_L}^{x_R} Q(x, t_1) dx - \frac{1}{\Delta x} \Delta t \left[ \frac{1}{\Delta t} \int_{t_1}^{t_2} F(Q(x_R, t)) dt - \frac{1}{\Delta t} \int_{t_1}^{t_2} F(Q(x_L, t)) dt \right]$$

$$\Rightarrow Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} [F_{i+1/2} - F_{i-1/2}]$$

## **Conservative schemes in 1D**

$$\partial_t Q + \partial_x F(Q) = 0$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} [F_{i+1/2} - F_{i-1/2}]$$

**Task: define numerical flux**

$$F_{i+1/2}$$

Basic property required: MONOTONICITY

**There are two approaches:**

**I: Upwind approach. Solve the Riemann problem**

$$\left. \begin{array}{l} \partial_t Q + \partial_x F(Q) = 0 \\ Q(x,0) = \begin{cases} Q_i^n & \text{if } x < 0 \\ Q_{i+1}^n & \text{if } x > 0 \end{cases} \end{array} \right\} \Rightarrow F_{i+1/2}$$

**II: Centred approach. The numerical flux is**

$$F_{i+1/2} = H(Q_i^n, Q_{i+1}^n)$$

## Properties required from 2-point flux

$$F_{i+1/2} = H(U, V)$$

Consistency:  $F_{i+1/2} = H(U, U) = F(U)$

Monotonicity:

$$f_{i+1/2} = h(q_i^n, q_{i+1}^n) \rightarrow q_i^{n+1} = L(\dots q_{i-1}^n, q_i^n, q_{i+1}^n \dots)$$

**Definition:** a monotone scheme satisfies

$$\frac{\partial}{\partial q_k^n} L(\dots q_{i-1}^n, q_i^n, q_{i+1}^n \dots) \geq 0 \quad \forall k$$

## Properties required from 2-point flux

**Remark:** for a linear scheme  $q_i^{n+1} = \sum_{k=-1}^r \beta_k q_{i+k}^n$

monotonicity requires positivity of coefficients:  $\beta_k \geq 0 \forall k$

**Theorem:** for a two-point flux, necessary conditions for monotonicity are

$$\frac{\partial}{\partial u} f_{i+1/2} = \frac{\partial}{\partial u} h(u, v) \geq 0; \quad \frac{\partial}{\partial v} f_{i+1/2} = \frac{\partial}{\partial v} h(u, v) \leq 0$$

# Classical centred numerical fluxes

The Lax-Friedrichs flux

$$F_{i+1/2}^{LF} = \frac{1}{2} \left( F(Q_i^n) + F(Q_{i+1}^n) \right) - \frac{1}{2} \frac{\Delta x}{\Delta t} \left( Q_{i+1}^n - Q_i^n \right)$$

Properties

1. Linearly stable for  $0 \leq |c| \leq 1$
2. Monotone for all CFL numbers in the stability range
3. Largest local truncation error of all monotone schemes

$c = \Delta t \lambda / \Delta x$  the Courant number

## Classical centred numerical fluxes, contin...

The Lax-Wendroff flux (2 versions)

$$F_{i+1/2}^{LW} = F(Q_{i+1/2}^{lw}), \quad Q_{i+1/2}^{lw} = \frac{1}{2} (Q_i^n + Q_{i+1}^n) - \frac{1}{2} \frac{\Delta t}{\Delta x} (F(Q_{i+1}^n) - F(Q_i^n))$$

$$F_{i+1/2}^{LW} = \frac{1}{2} (F(Q_i^n) + (Q_{i+1}^n)) - \frac{1}{2} \frac{\Delta t}{\Delta x} A_{i+1/2} (F(Q_{i+1}^n) - F(Q_i^n))$$

### Properties

1. Linearly stable for  $0 \leq |c| \leq 1$
2. Non-monotone (oscillatory)
3. Second-order accurate in space and time

## Classical centred numerical fluxes, contin...

The Godunov centred flux (1961)

$$F_{i+1/2}^{GC} = F(Q_{i+1/2}^{gc}), \quad Q_{i+1/2}^{gc} = \frac{1}{2} (Q_i^n + Q_{i+1}^n) - \frac{\Delta t}{\Delta x} (F(Q_{i+1}^n) - F(Q_i^n))$$

### Properties

1. Linearly stable for  $0 \leq |c| \leq \frac{1}{2} \sqrt{2}$
2. Monotone for  $\frac{1}{2} \leq |c| \leq \frac{1}{2} \sqrt{2}$
3. Non-monotone for  $0 \leq |c| \leq \frac{1}{2}$

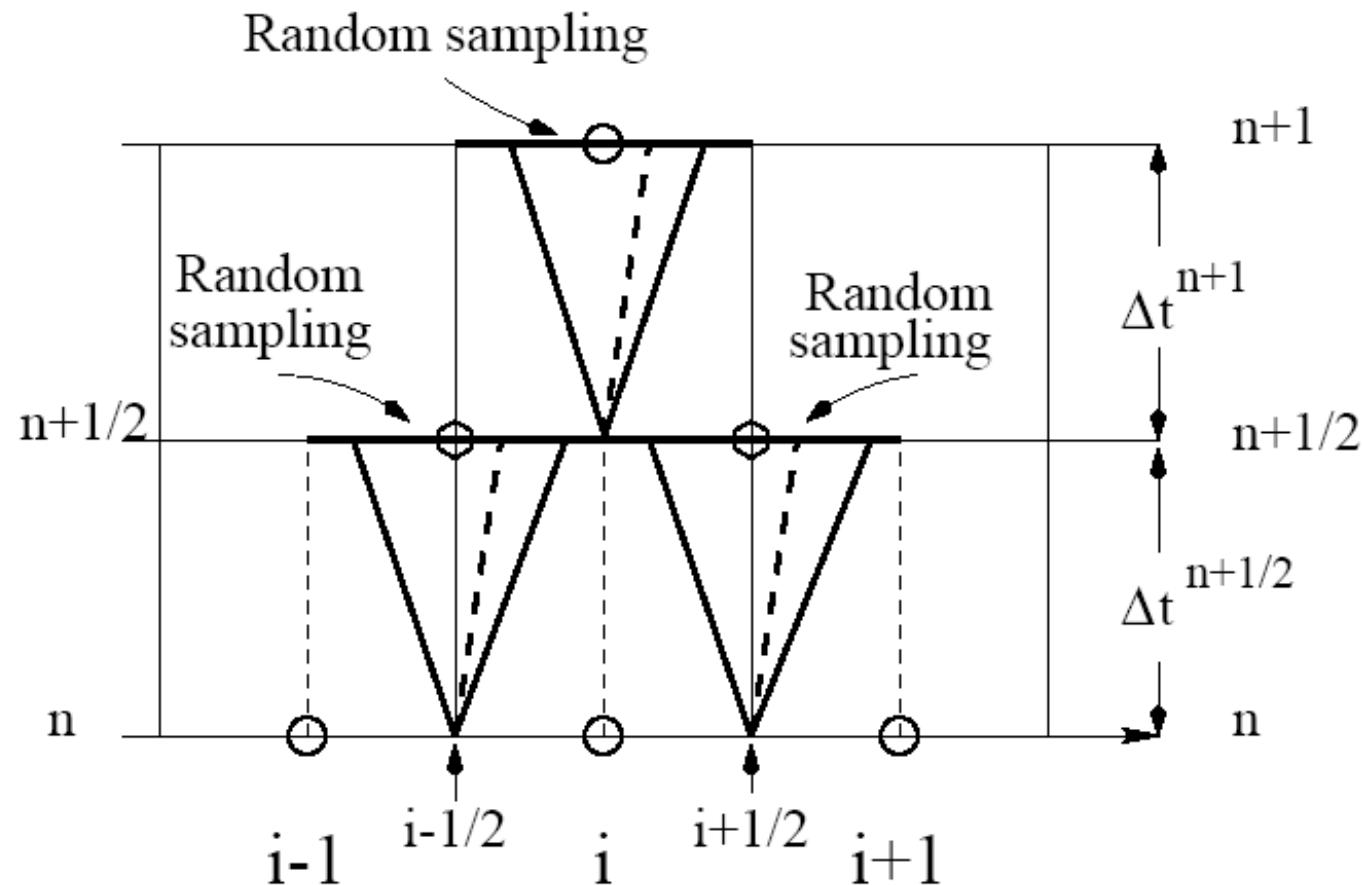
# The FORCE flux (First ORder CEntred)

*Toro E F.*

*On Glimm-related schemes for conservation laws.*

*Technical Report MMU-9602, Department of Mathematics and  
Physics, Manchester Metropolitan University, 1996, UK*

## Glimm's method on a staggered mesh



Recall the integral form of the conservation laws

$$\partial_t Q + \partial_x F(Q) = 0$$

in a control volume  $[x_L, x_R] \times [t_1, t_2]$

$$\frac{1}{\Delta x} \int_{x_L}^{x_R} Q(x, t_2) dx = \frac{1}{\Delta x} \int_{x_L}^{x_R} Q(x, t_1) dx - \frac{1}{\Delta x} \Delta t \left[ \frac{1}{\Delta t} \int_{t_1}^{t_2} F(Q(x_R, t)) dt - \frac{1}{\Delta t} \int_{t_1}^{t_2} F(Q(x_L, t)) dt \right]$$

## Step I

$$Q_{i+1/2}^{n+1/2} = \frac{1}{2} \left( Q_i^n + Q_{i+1}^n \right) - \frac{1}{2} \frac{\Delta t}{\Delta x} \left( F(Q_{i+1}^n) - F(Q_i^n) \right)$$

$$Q_{i-1/2}^{n+1/2} = \frac{1}{2} \left( Q_{i-1}^n + Q_i^n \right) - \frac{1}{2} \frac{\Delta t}{\Delta x} \left( F(Q_i^n) - F(Q_{i-1}^n) \right)$$

## Step II

$$Q_i^{n+1} = \frac{1}{2} \left( Q_{i-1/2}^{n+1/2} + Q_{i+1/2}^{n+1/2} \right) - \frac{1}{2} \frac{\Delta t}{\Delta x} \left( F(Q_{i+1/2}^{n+1/2}) - F(Q_{i-1/2}^{n+1/2}) \right)$$

Question: can we write

$$Q_i^{n+1} = \frac{1}{2} \left( Q_{i-1/2}^{n+1/2} + Q_{i+1/2}^{n+1/2} \right) - \frac{1}{2} \frac{\Delta t}{\Delta x} \left( F(Q_{i+1/2}^{n+1/2}) - F(Q_{i-1/2}^{n+1/2}) \right)$$

as a one-step conservative method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} \left( F_{i+1/2}^{\text{force}} - F_{i-1/2}^{\text{force}} \right)$$

with a given numerical flux

$$F_{i+1/2}^{\text{force}}$$

Answer: YES

The numerical flux is

$$F_{i+1/2}^{\text{force}} = \frac{1}{4} \left[ F_i^n + 2F(Q_{i+1/2}^{n+1/2}) + F_{i+1}^n - \frac{\Delta x}{\Delta t} (Q_{i+1}^n - Q_i^n) \right]$$

But recall

$$F_{i+1/2}^{\text{LW}} = F(Q_{i+1/2}^{\text{lw}}), \quad Q_{i+1/2}^{\text{lw}} = \frac{1}{2} (Q_i^n + Q_{i+1}^n) - \frac{1}{2} \frac{\Delta t}{\Delta x} (F(Q_{i+1}^n) - F(Q_i^n))$$

$$F_{i+1/2}^{\text{LF}} = \frac{1}{2} (F(Q_i^n) + F(Q_{i+1}^n)) - \frac{1}{2} \frac{\Delta x}{\Delta t} (Q_{i+1}^n - Q_i^n)$$

The numerical flux is in fact

$$F_{i+1/2}^{\text{force}} = \frac{1}{2}(F_{i+1/2}^{\text{LW}} + F_{i+1/2}^{\text{LF}})$$

with

$$F_{i+1/2}^{\text{LW}} = F(Q_{i+1/2}^{\text{lw}}), \quad Q_{i+1/2}^{\text{lw}} = \frac{1}{2}(Q_i^n + Q_{i+1}^n) - \frac{1}{2} \frac{\Delta t}{\Delta x} (F(Q_{i+1}^n) - F(Q_i^n))$$

$$F_{i+1/2}^{\text{LF}} = \frac{1}{2}(F(Q_i^n) + F(Q_{i+1}^n)) - \frac{1}{2} \frac{\Delta x}{\Delta t} (Q_{i+1}^n - Q_i^n)$$

## Properties of the FORCE scheme

$$\partial_t q + \lambda \partial_x q = 0$$

$$q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta x} [f_{i+1/2} - f_{i-1/2}]$$

$$f_{i+1/2}^{\text{force}} = \frac{(1+c)^2}{4c} (\lambda q_i^n) + \frac{(1-c)^2}{4c} (\lambda q_{i+1}^n)$$

$$q_i^{n+1} = b_{-1} q_{i-1}^n + b_0 q_i^n + b_1 q_{i+1}^n$$

$$b_{-1} = \frac{1}{4}(1+c)^2 \quad b_0 = \frac{1}{2}(1-c^2) \quad b_1 = \frac{1}{4}(1-c)^2$$

## Properties of the FORCE scheme, cont.

*Stable*     $0 \leq |c| \leq 1$

*Monotone*

*Modified equation*  $\partial_t q + \lambda \partial_x q = \alpha_{force} \partial_x^{(2)} q$

$$\alpha_{force} = \frac{1}{4} \lambda \Delta x \left( \frac{1 - c^2}{c} \right) = \frac{1}{2} \alpha_{lf}$$

Proof of convergence of FORCE scheme in:

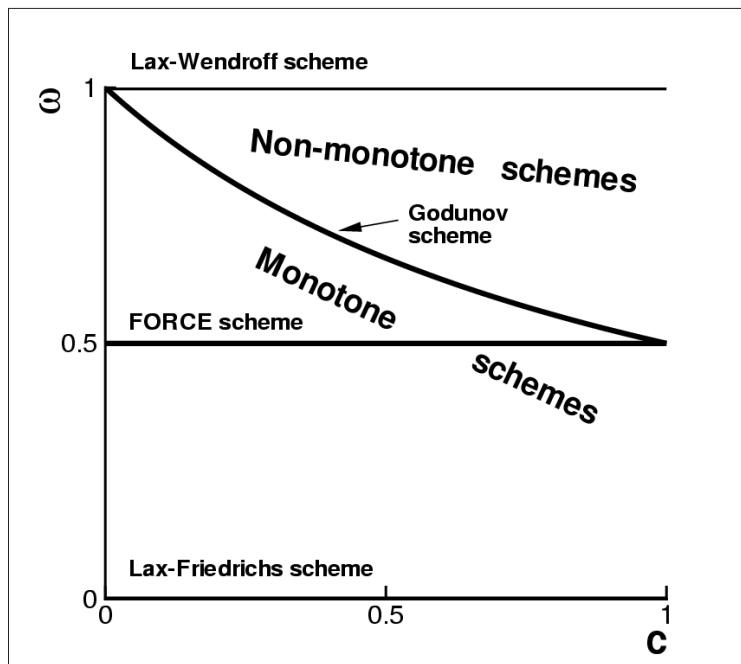
*Chen C Q and Toro E F.*

*Centred schemes for non-linear hyperbolic equations.*

*Journal of Hyperbolic Differential Equations. Vol. 1 (1), pp 531-566, 2004.*

# The FORCE flux for the scalar case: more general averaging.

$$F_{i+1/2}^\omega = \omega F_{i+1/2}^{LW} + (1 - \omega) F_{i+1/2}^{LF}, \quad 0 < \omega < 1$$



Special cases:

$\omega = 0$  (Lax - Friedrichs)

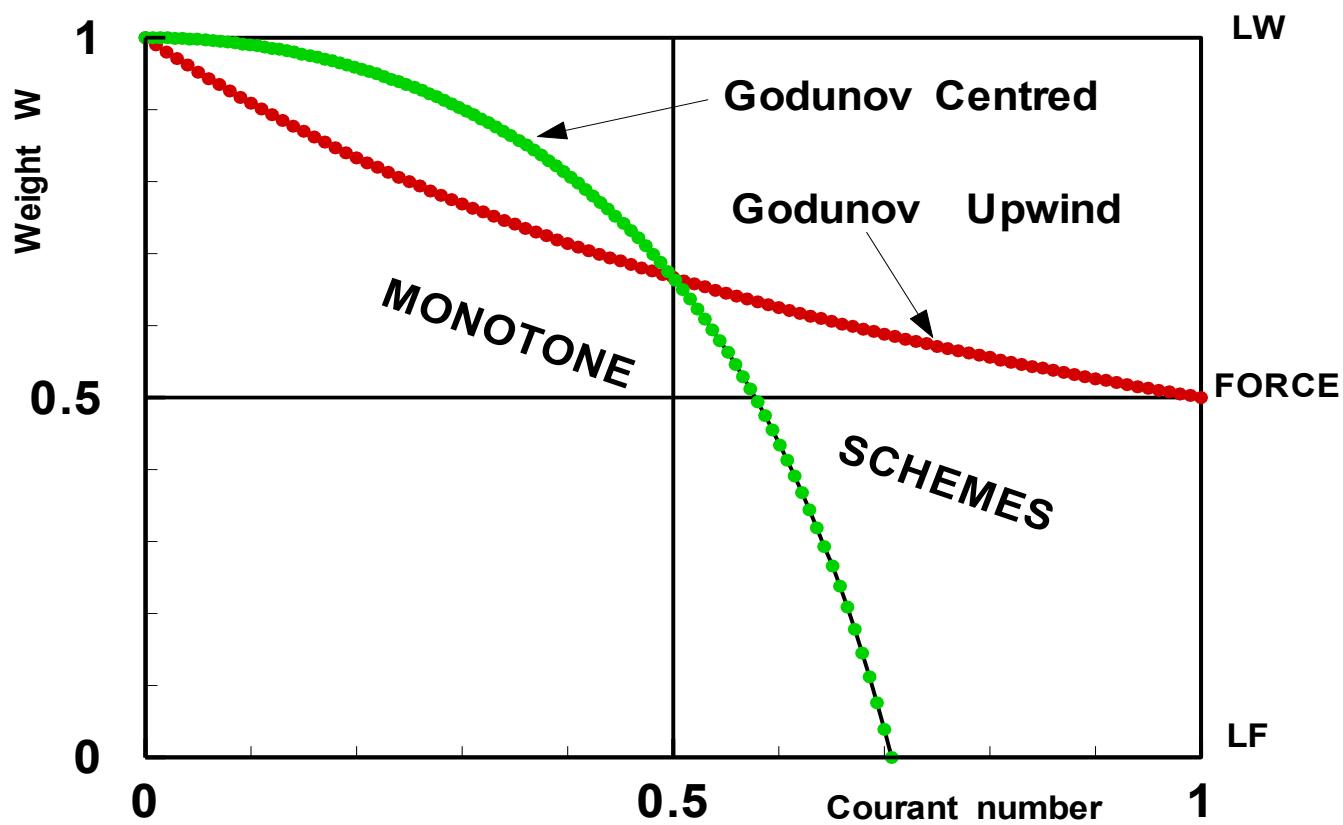
$\omega = 1$  (Lax - Wendroff)

$\omega = 1/2$  (FORCE)

$$\omega = \frac{1}{1+c} \text{ (GFORCE)}$$

## Monotonocity

$$0 \leq \omega \leq \omega_{\max} \equiv \frac{1}{1+|c|} \quad \frac{1}{2} \leq \omega_{\max} \equiv \frac{1}{1+|c|} \leq 1$$



# *FORCE's friends and relatives*

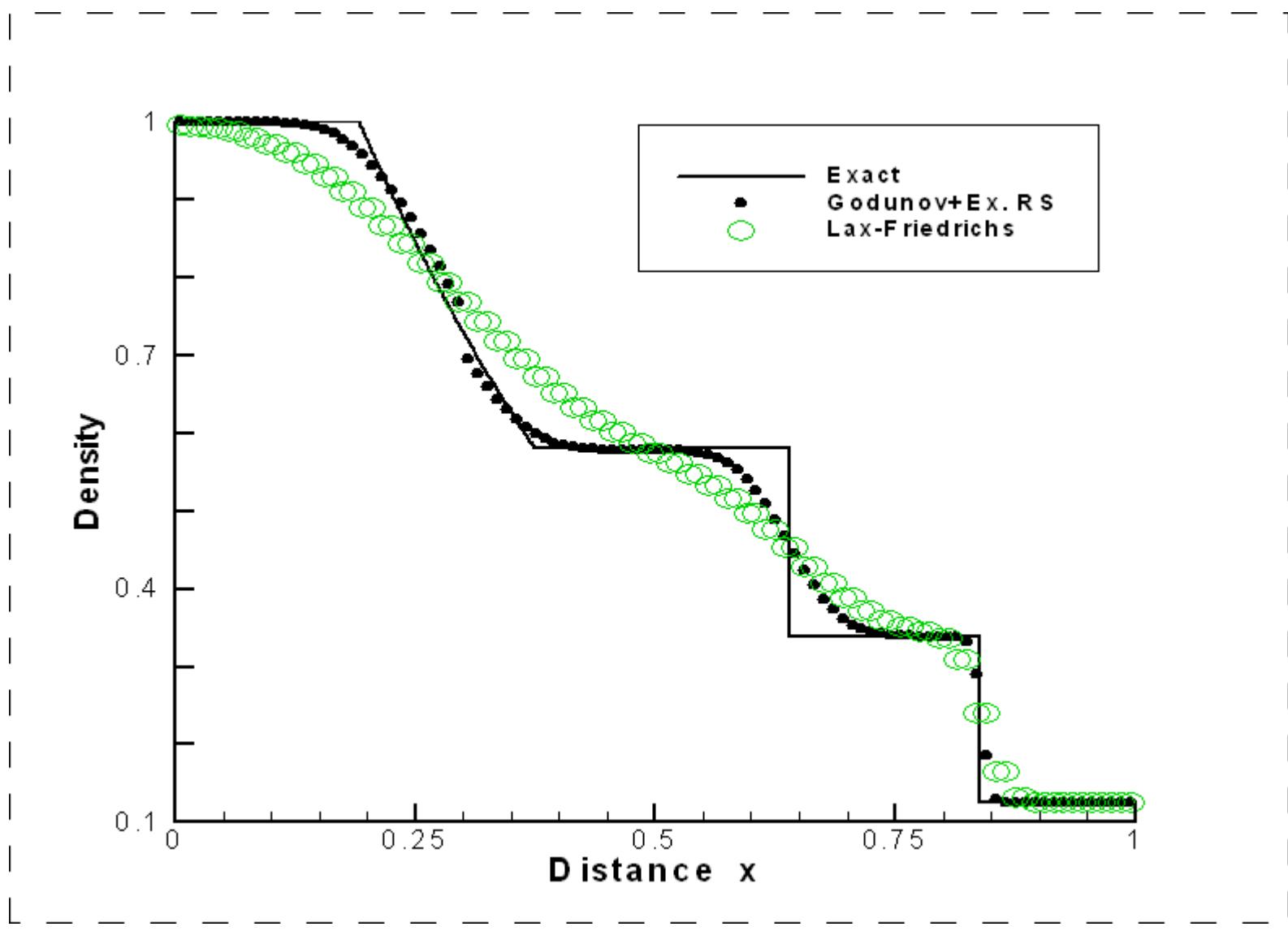
- The composite schemes of Liska and Wendroff (friend)

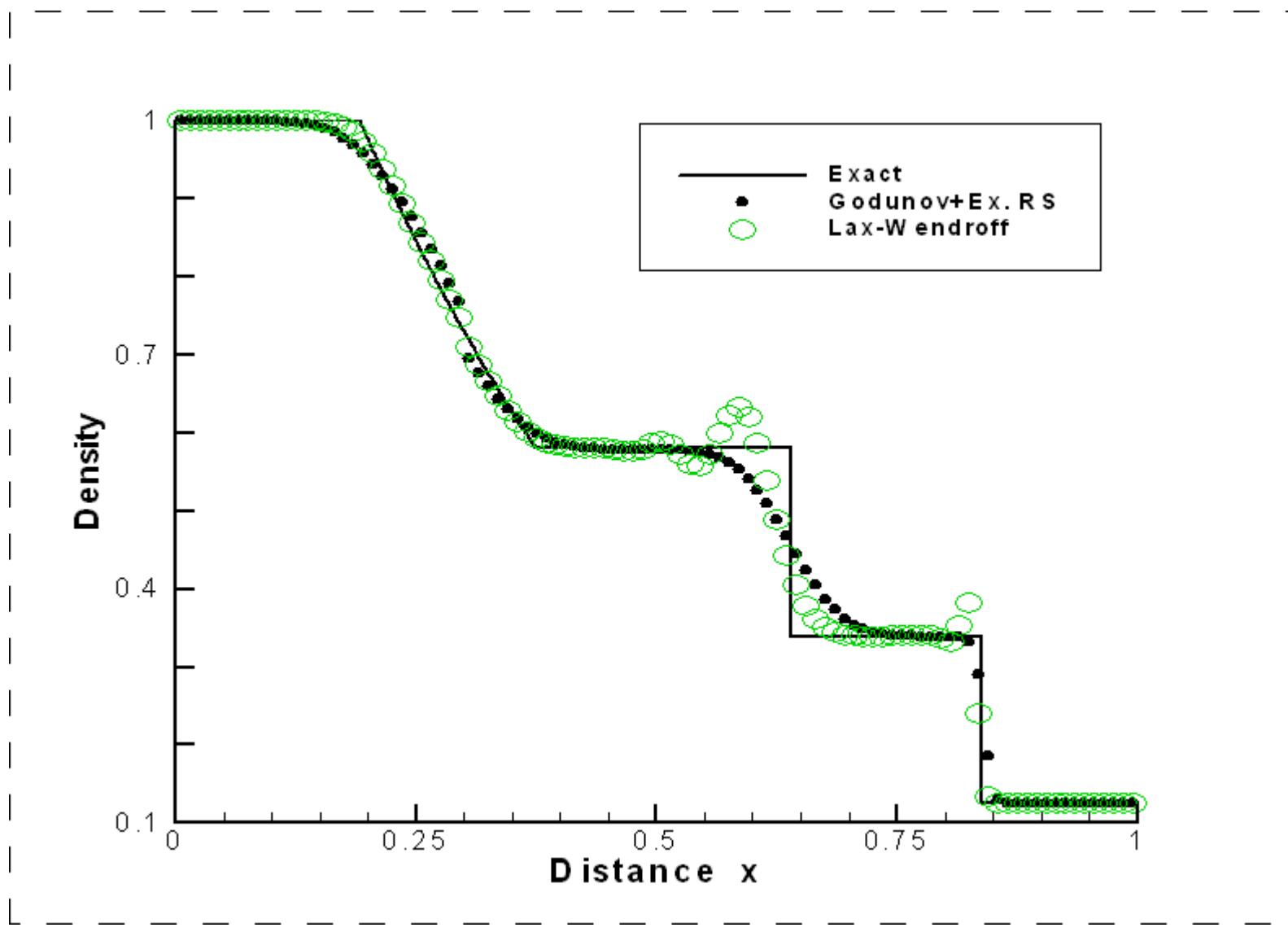
*Liska R and Wendroff B. Composite schemes for conservation laws. SIAM J. Numerical Analysis, Vol. 35, pp 2250-2271, 1998*

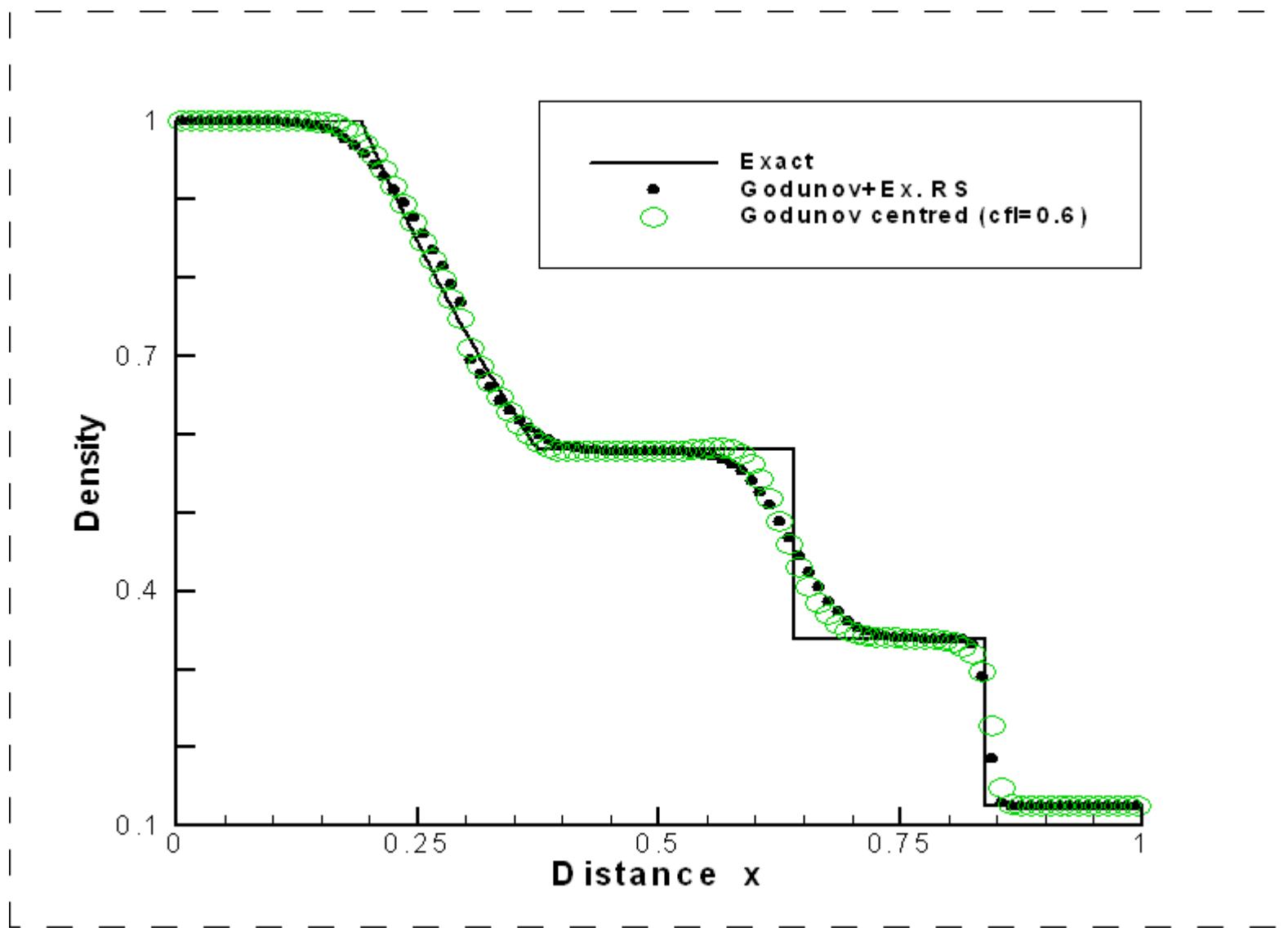
- The centred scheme of Nessyahu and Tadmor (relative)

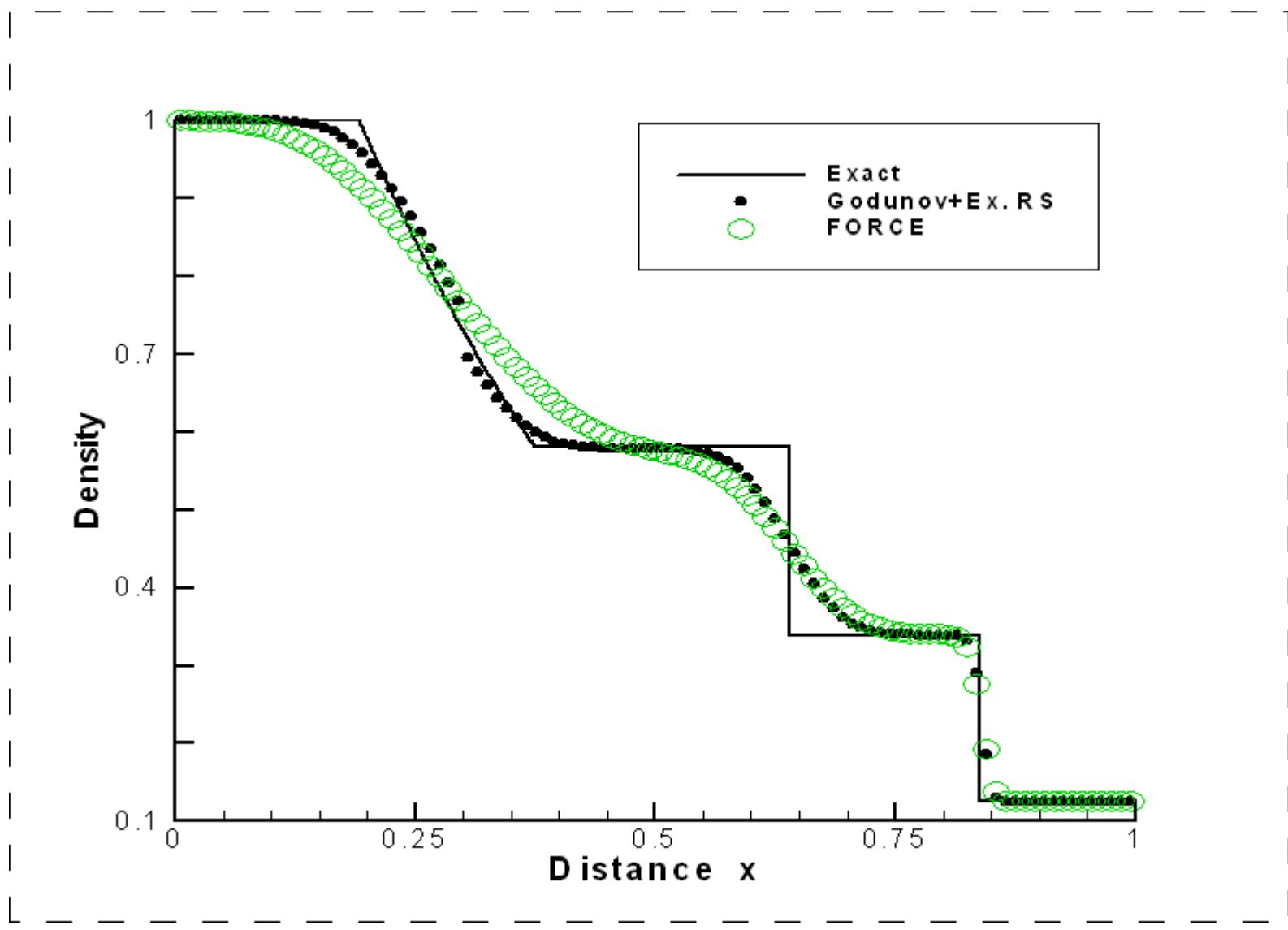
*Non-oscillatory central differencing for hyperbolic conservation Laws. J. Computational Physics, Vol 87, pp 408-463, 1990.*

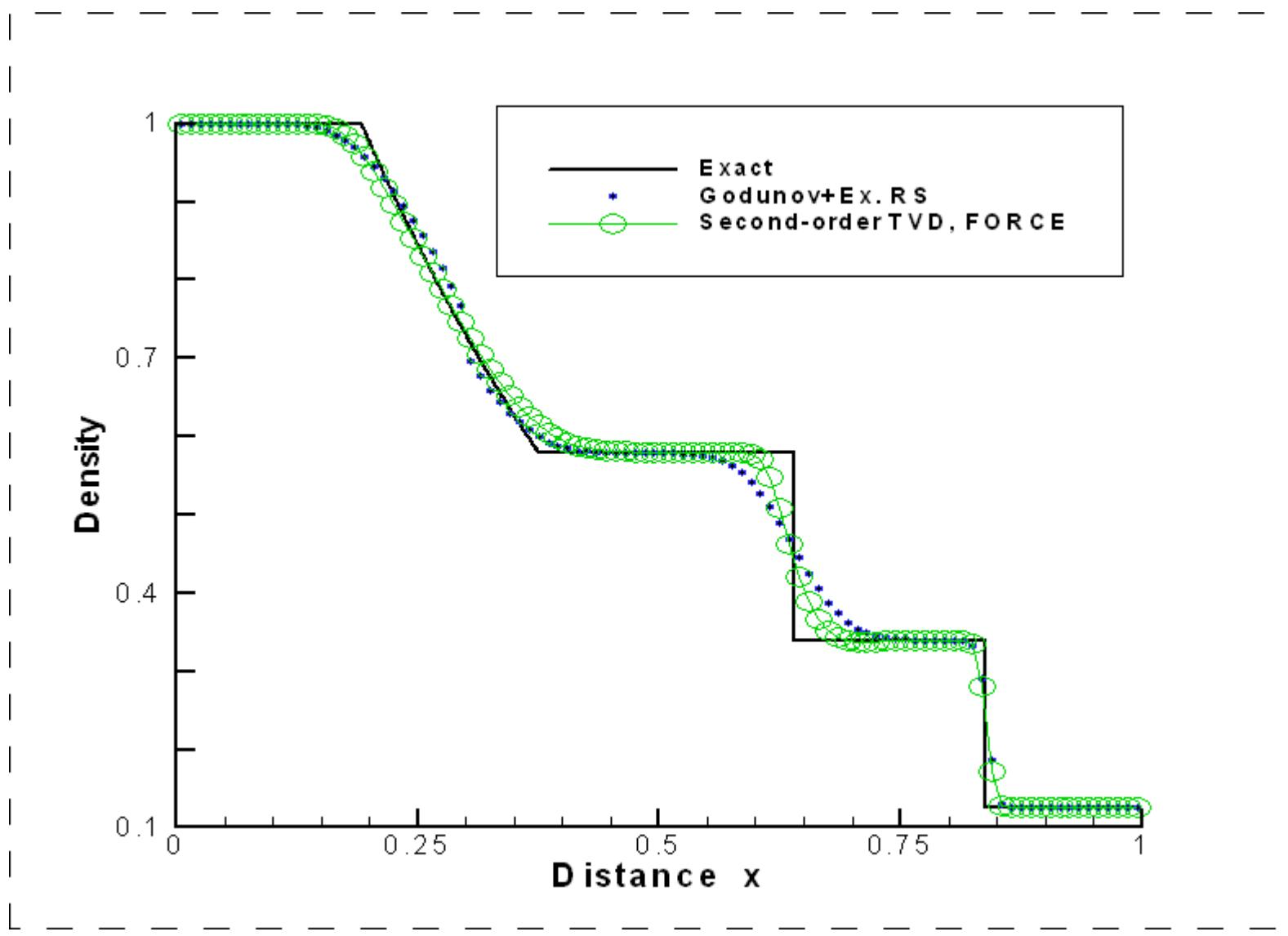
# **Numerical results for 1D Euler equations**











# *How about extensions of FORCE ?*

- High-order non-oscillatory extensions
- Source terms
- Multiple space dimensions
- Unstructured meshes

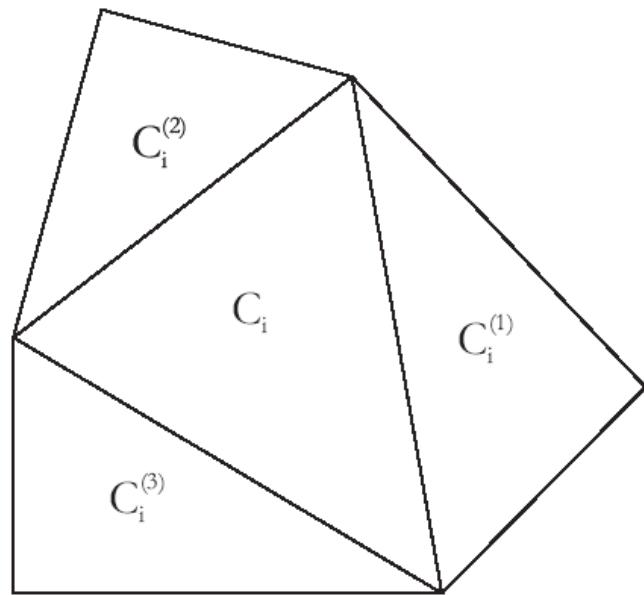
# *FORCE schemes on unstructured meshes*

Toro E F, Hidalgo A and Dumbser M.

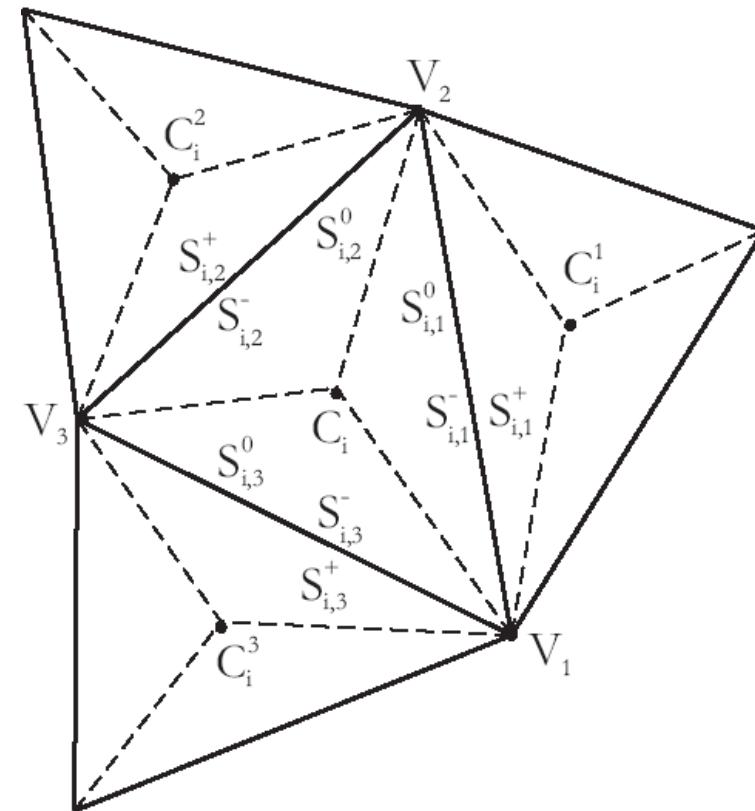
*FORCE schemes on unstructured meshes I:  
Conservative hyperbolic systems.*

(Journal of Computational Physics, Vol. 228, pp 3368-3389, 2009)

# Illustration in 2D



Triangular primary mesh



Primary and secondary mesh

## Step I

Initial condition: integral averages at time n     $\bar{Q}_i^n$

Averaging operator applied on edge-based control volume gives

$$Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{\bar{Q}_i^n V_i^- + \bar{Q}_j^n V_j^+}{V_j^- + V_j^+} - \frac{1}{2} \frac{\Delta t S_j}{V_j^- + V_j^+} (\underline{\underline{F}}(\bar{Q}_j^n) - \underline{\underline{F}}(\bar{Q}_i^n)) \cdot \vec{n}_j$$

$$\begin{matrix} F \\ = \end{matrix}$$

$V_j^-$       Portion of j edge-base volume inside cell i

$V_j^+$       Portion of j edge-base volume outside cell i

$S_j$       Area of face j (between cells i and j)

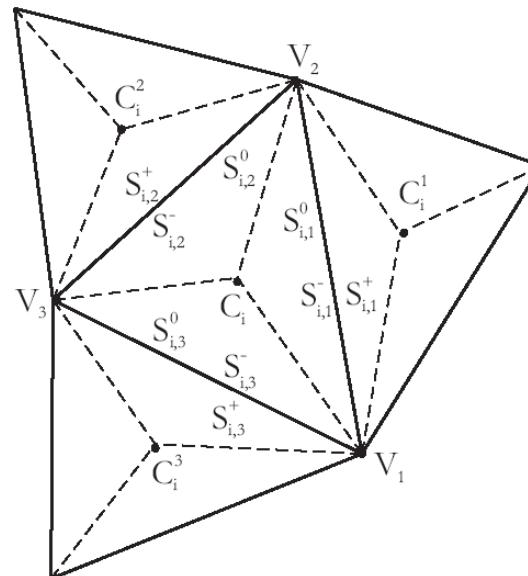
$n_j$       Unit outward normal vector to of face j

## Step II

Initial condition: integral averages at time  $n+1/2$   $\mathbf{Q}_{j+1/2}^{n+1/2}$

Averaging operator applied on primary mesh gives

$$\mathbf{Q}_i^{n+1} = \frac{1}{|T_i|} \sum_{j=1}^{n_f} \left( \mathbf{Q}_{j+\frac{1}{2}}^{n+\frac{1}{2}} V_j^- - \frac{1}{2} \Delta t S_j \underline{\underline{\mathbf{F}}} \left( \mathbf{Q}_{j+\frac{1}{2}}^{n+\frac{1}{2}} \right) \cdot \vec{n}_j \right)$$



## Step III: one-step conservative scheme

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{|T_i|} \sum_{j=1}^{n_f} S_j \underline{\underline{\mathbf{F}}}{}^{\text{FORCE}\alpha}_{j+\frac{1}{2}} \cdot \vec{n}_j$$

$$\underline{\underline{\mathbf{F}}}{}^{\text{FORCE}\alpha}_{j+\frac{1}{2}} = \frac{1}{2} \left( \underline{\underline{\mathbf{F}}}{}^{LW\alpha}_{j+\frac{1}{2}} (\mathbf{Q}_i^n, \mathbf{Q}_j^n) + \underline{\underline{\mathbf{F}}}{}^{LF\alpha}_{j+\frac{1}{2}} (\mathbf{Q}_i^n, \mathbf{Q}_j^n) \right)$$

$$\underline{\underline{\mathbf{F}}}{}^{LW\alpha}_{j+\frac{1}{2}} (\mathbf{Q}_i^n, \mathbf{Q}_j^n) = \underline{\underline{\mathbf{F}}}{} \left( \mathbf{Q}_{j+\frac{1}{2}}^{n+\frac{1}{2}} \right),$$

$$\mathbf{Q}_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{\mathbf{Q}_i^n V_i^- + \mathbf{Q}_j^n V_j^+}{V_j^- + V_j^+} - \frac{1}{2} \frac{\Delta t S_j}{V_j^- + V_j^+} \left( \underline{\underline{\mathbf{F}}}{} (\mathbf{Q}_j^n) - \underline{\underline{\mathbf{F}}}{} (\mathbf{Q}_i^n) \right) \cdot \vec{n}_j$$

$$\underline{\underline{\mathbf{F}}}{}^{LF\alpha}_{j+\frac{1}{2}} (\mathbf{Q}_i^n, \mathbf{Q}_j^n) = \frac{V_j^- \underline{\underline{\mathbf{F}}}{} (\mathbf{Q}_j^n) + V_j^+ \underline{\underline{\mathbf{F}}}{} (\mathbf{Q}_i^n)}{V_j^- + V_j^+} - \frac{V_j^- V_j^+}{V_j^- + V_j^+} \frac{2}{\Delta t S_j} (\mathbf{Q}_j^n - \mathbf{Q}_i^n) \vec{n}_j^T$$

# The FORCE flux in $\alpha$ space dimensions on Cartesian meshes

$$F_{i+1/2,j,k}^{\text{force}\alpha} = \frac{1}{2}(F_{i+1/2,j,k}^{\text{lwa}} + F_{i+1/2,j,k}^{\text{lfa}})$$

Lax-Wendroff type flux

$$F_{i+1/2,j}^{\text{lwa}} = F(Q_{i+1/2,j}^{\text{lwa}}),$$

$$Q_{i+1/2,j,k}^{\text{lwa}} = \frac{1}{2}(Q_{i,j,k}^n + Q_{i+1,j,k}^n) - \frac{1}{2} \frac{\alpha \Delta t}{\Delta x} (F(Q_{i+1,j,k}^n) - F(Q_{i,j,k}^n))$$

Lax-Friedrichs type flux

$$F_{i+1/2,j,k}^{\text{lfa}} = \frac{1}{2}(F(Q_{i,j,k}^n) + F(Q_{i+1,j,k}^n)) - \frac{1}{2} \frac{\Delta x}{\alpha \Delta t} (Q_{i+1,j,k}^n - Q_{i,j,k}^n)$$

# FORCE-type fluxes

$$\underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{\text{GFORCE}\alpha} = \omega \underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{LW\alpha} (\mathbf{Q}_i^n, \mathbf{Q}_j^n) + (1 - \omega) \underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{LF\alpha} (\mathbf{Q}_i^n, \mathbf{Q}_j^n)$$

Stability and monotonicity results

$\omega$	1D	2D	3D
$0 \leq \omega < \frac{1}{2}$	$ c  \leq \frac{1}{\alpha}$	$ c_x ,  c_y  \leq \frac{1}{2}$	$ c_x ,  c_y ,  c_z  \leq \frac{1}{3}$
$\omega = \frac{1}{2}$	$ c  \leq \frac{\sqrt{2\alpha-1}}{\alpha}$	$c_x^2 + c_y^2 \leq \frac{1}{2}$	$c_x^2 + c_y^2 + c_z^2 \leq \frac{1}{3}$
$\frac{1}{2} < \omega < 1$	$ c  \leq \left  \frac{-1+\omega}{\omega\alpha} \right $	$ c_x ,  c_y  \leq \left  \frac{-1+\omega}{2\omega} \right $	$ c_x ,  c_y ,  c_z  \leq \left  \frac{-1+\omega}{3\omega} \right $

# One-dimensional interpretation

$$\partial_t Q + \partial_x F(Q) = 0$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} [F_{i+1/2} - F_{i-1/2}]$$

$$F_{i+1/2}^{\text{force}\alpha} = \frac{1}{2} (F_{i+1/2}^{\text{lw}\alpha} + F_{i+1/2}^{\text{lfa}})$$

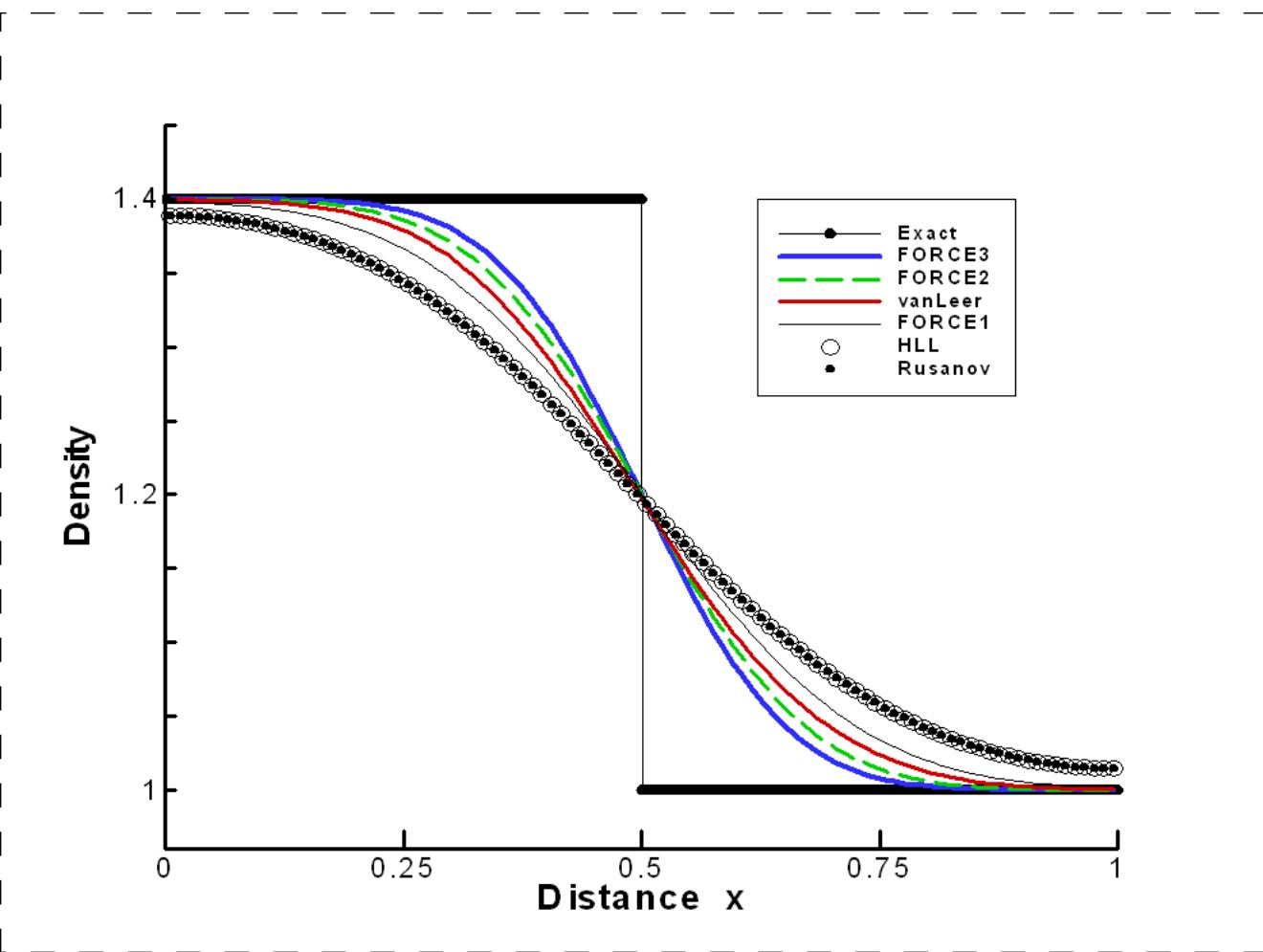
$$F_{i+1/2}^{\text{lw}\alpha} = F(Q_{i+1/2}^{\text{lw}\alpha})$$

$$Q_{i+1/2}^{\text{lw}\alpha} = \frac{1}{2} (Q_i^n + Q_{i+1}^n) - \frac{1}{2} \frac{\alpha \Delta t}{\Delta x} (F(Q_{i+1}^n) - F(Q_i^n))$$

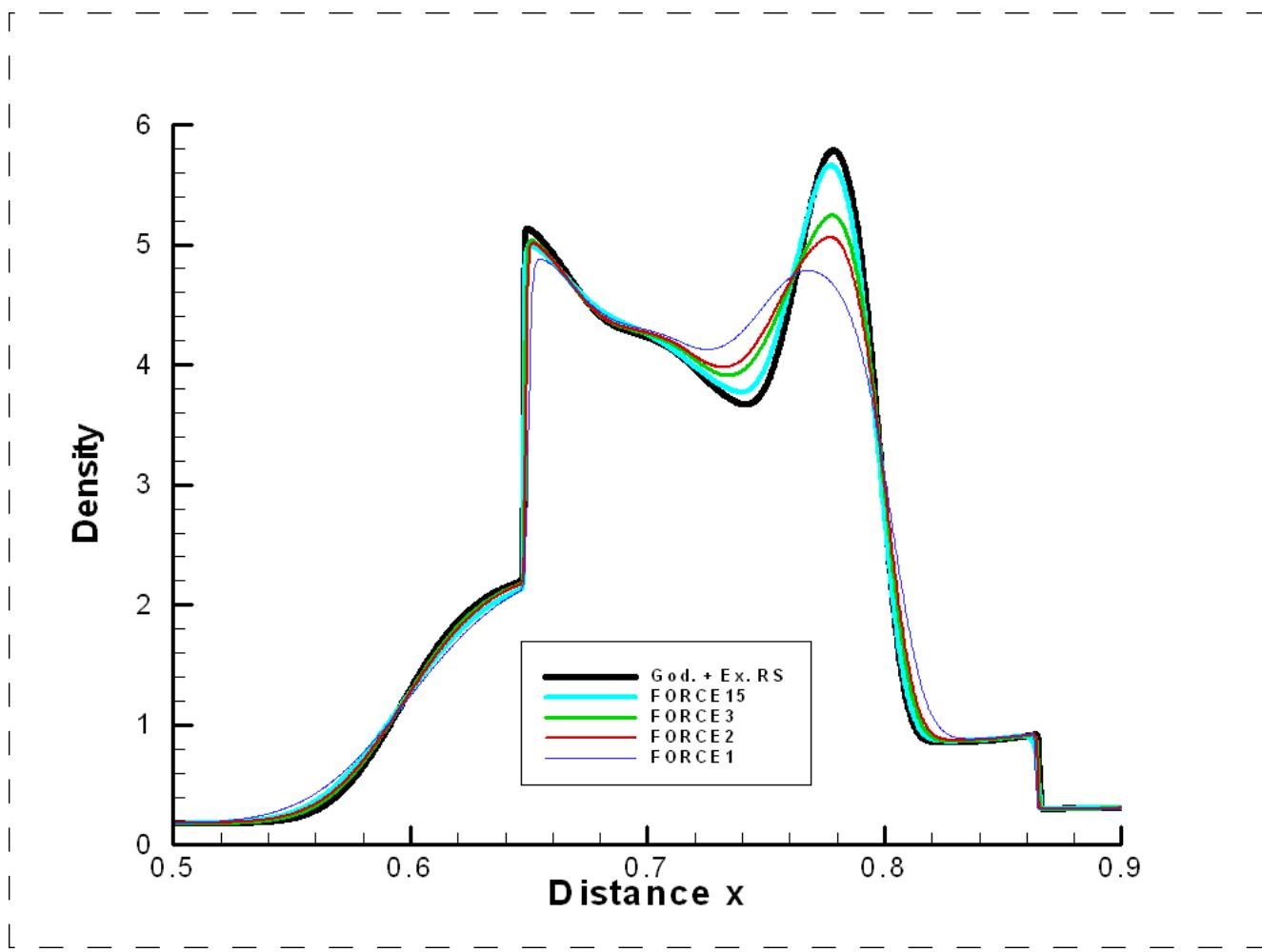
$$F_{i+1/2}^{\text{lfa}} = \frac{1}{2} (F(Q_i^n) + F(Q_{i+1}^n)) - \frac{1}{2} \frac{\Delta x}{\alpha \Delta t} (Q_{i+1}^n - Q_i^n)$$

$\alpha$ : parameter

# Numerical results for the 1D Euler equations



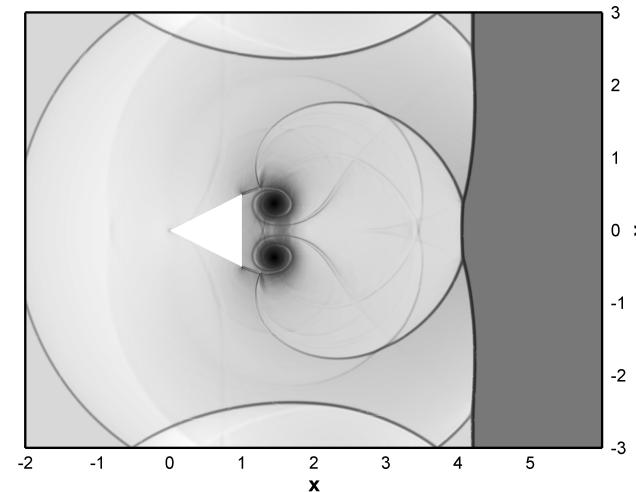
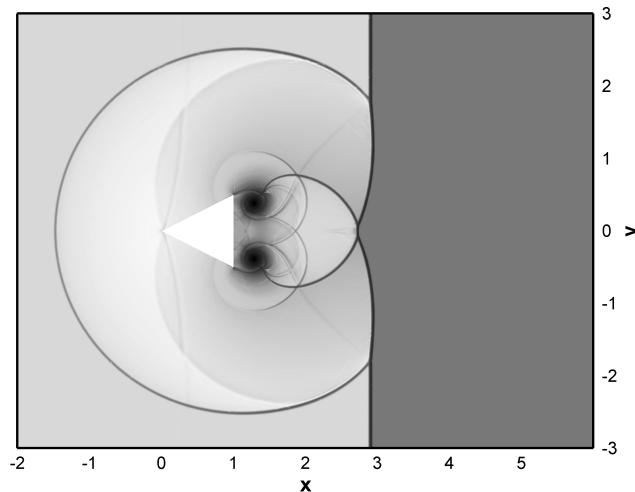
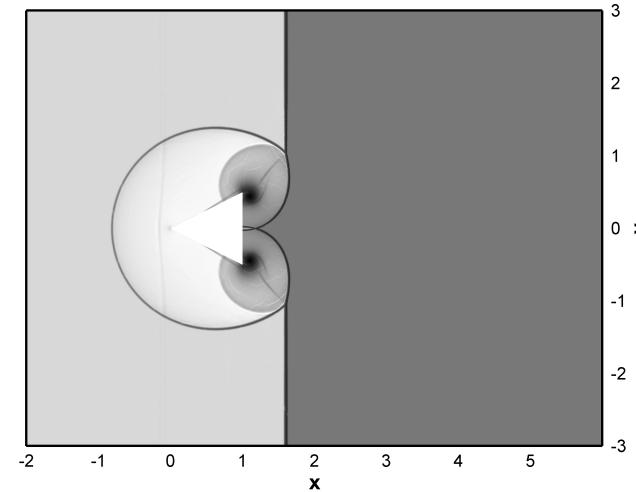
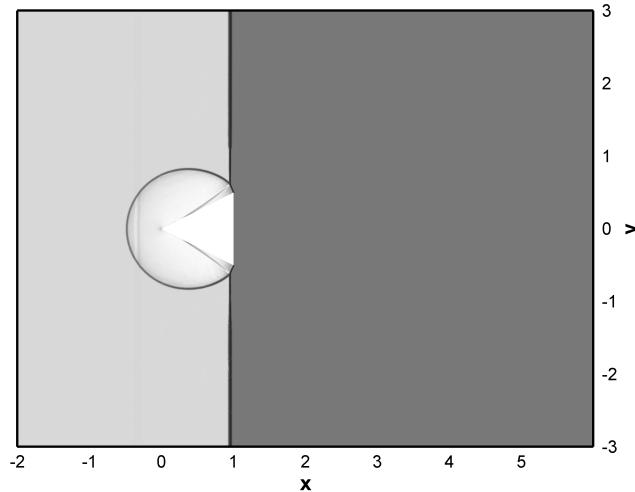
# Numerical results for the 1D Euler equations



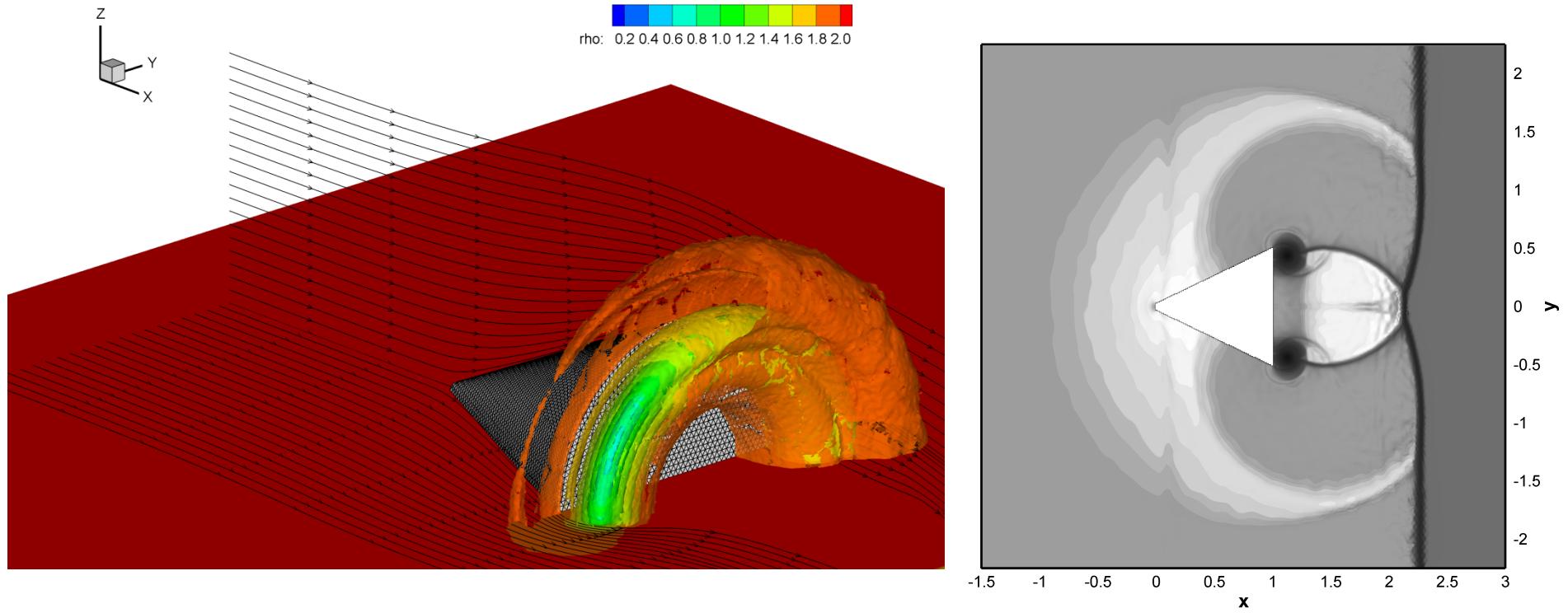
*Numerical results:*

*Euler equations in 2D and 3D*

## *2D Euler equations: reflection from triangle*



## *3D Euler equations: reflection from cone*



## *Numerical results:*

*The Baer-Nunziato equations in  
2D and 3D*

## *Application of ADER to the 3D Baer-Nunziato equations*

$$\left. \begin{array}{l} \frac{\partial}{\partial t} (\phi_1 \rho_1) + \nabla \cdot (\phi_1 \rho_1 \mathbf{u}_1) = 0, \\ \frac{\partial}{\partial t} (\phi_1 \rho_1 \mathbf{u}_1) + \nabla \cdot (\phi_1 \rho_1 \mathbf{u}_1 \otimes \mathbf{u}_1) + \nabla \phi_1 p_1 = p_I \nabla \phi_1 + \lambda (\mathbf{u}_2 - \mathbf{u}_1), \\ \frac{\partial}{\partial t} (\phi_1 \rho_1 E_1) + \nabla \cdot ((\phi_1 \rho_1 E_1 + \phi_1 p_1) \mathbf{u}_1) = -p_I \partial_t \phi_1 + \lambda \mathbf{u}_I \cdot (\mathbf{u}_2 - \mathbf{u}_1), \\ \frac{\partial}{\partial t} (\phi_2 \rho_2) + \nabla \cdot (\phi_2 \rho_2 \mathbf{u}_2) = 0, \\ \frac{\partial}{\partial t} (\phi_2 \rho_2 \mathbf{u}_2) + \nabla \cdot (\phi_2 \rho_2 \mathbf{u}_2 \otimes \mathbf{u}_2) + \nabla \phi_2 p_2 = p_I \nabla \phi_2 - \lambda (\mathbf{u}_2 - \mathbf{u}_1), \\ \frac{\partial}{\partial t} (\phi_2 \rho_2 E_2) + \nabla \cdot ((\phi_2 \rho_2 E_2 + \phi_2 p_2) \mathbf{u}_2) = p_I \partial_t \phi_1 - \lambda \mathbf{u}_I \cdot (\mathbf{u}_2 - \mathbf{u}_1), \\ \frac{\partial}{\partial t} \phi_1 + \mathbf{u}_I \nabla \phi_1 = 0. \end{array} \right\} \quad (54)$$

11 non-linear hyperbolic PDES  
stiff source terms: relaxation terms

## **EXTENSION TO NONCONSERVATIVE SYSTEMS:**

Manuel Castro, Carlos Pares, Eleuterio Toro, Michael Dumbser and  
Arturo Hidalgo.

*FORCE schemes on unstructured meshes II: non-conservative  
hyperbolic systems.*

Computer Methods in Applied Mechanics and Engineering.  
Vol. 199, Issue 9-12, pages 625-647, 2010.

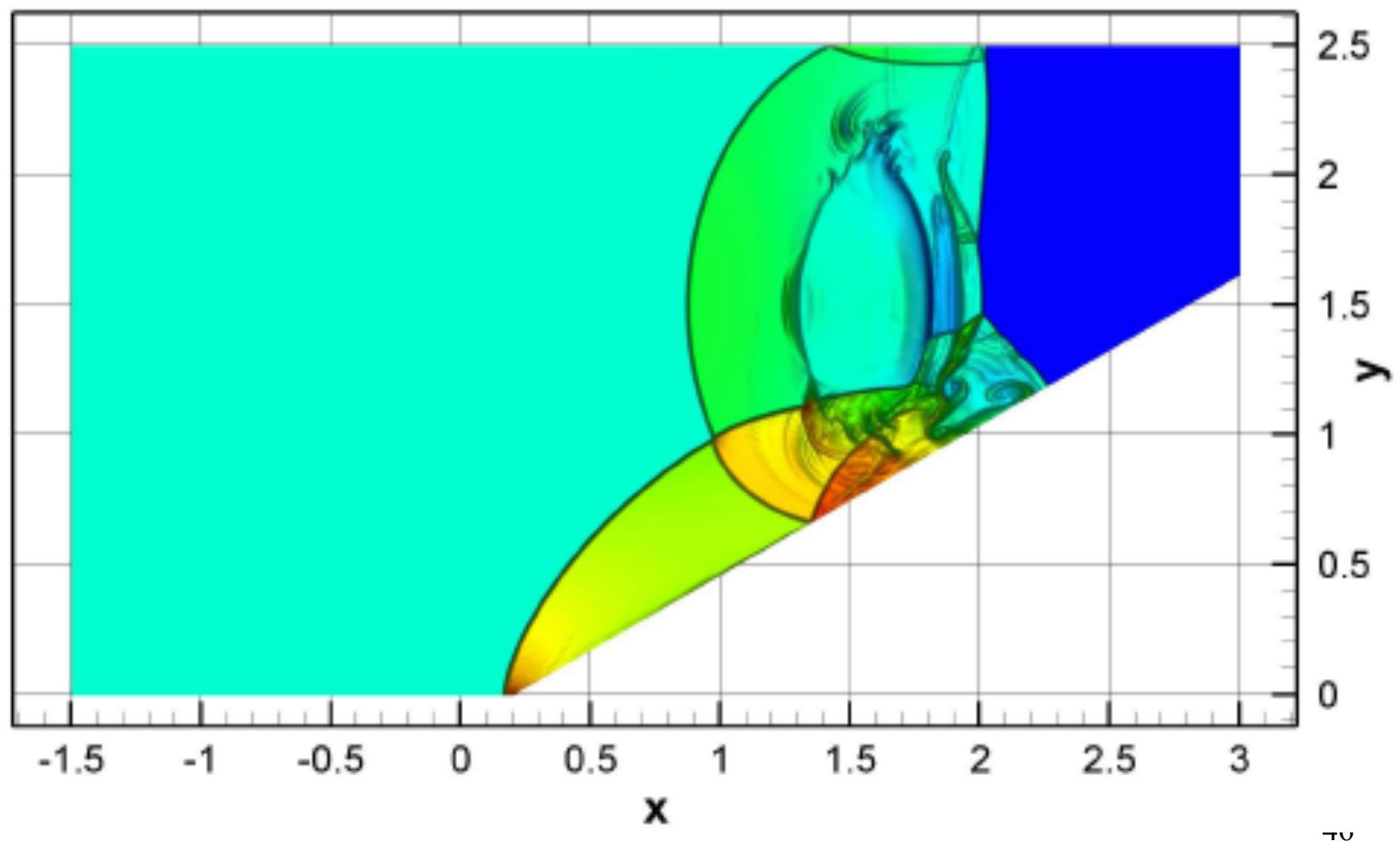
**Also published (NI09005-NPA) in pre-print series of the  
Newton Institute for Mathematical Sciences  
University of Cambridge, UK.**

**It can be downloaded from**  
<http://www.newton.ac.uk/preprints2009.html>

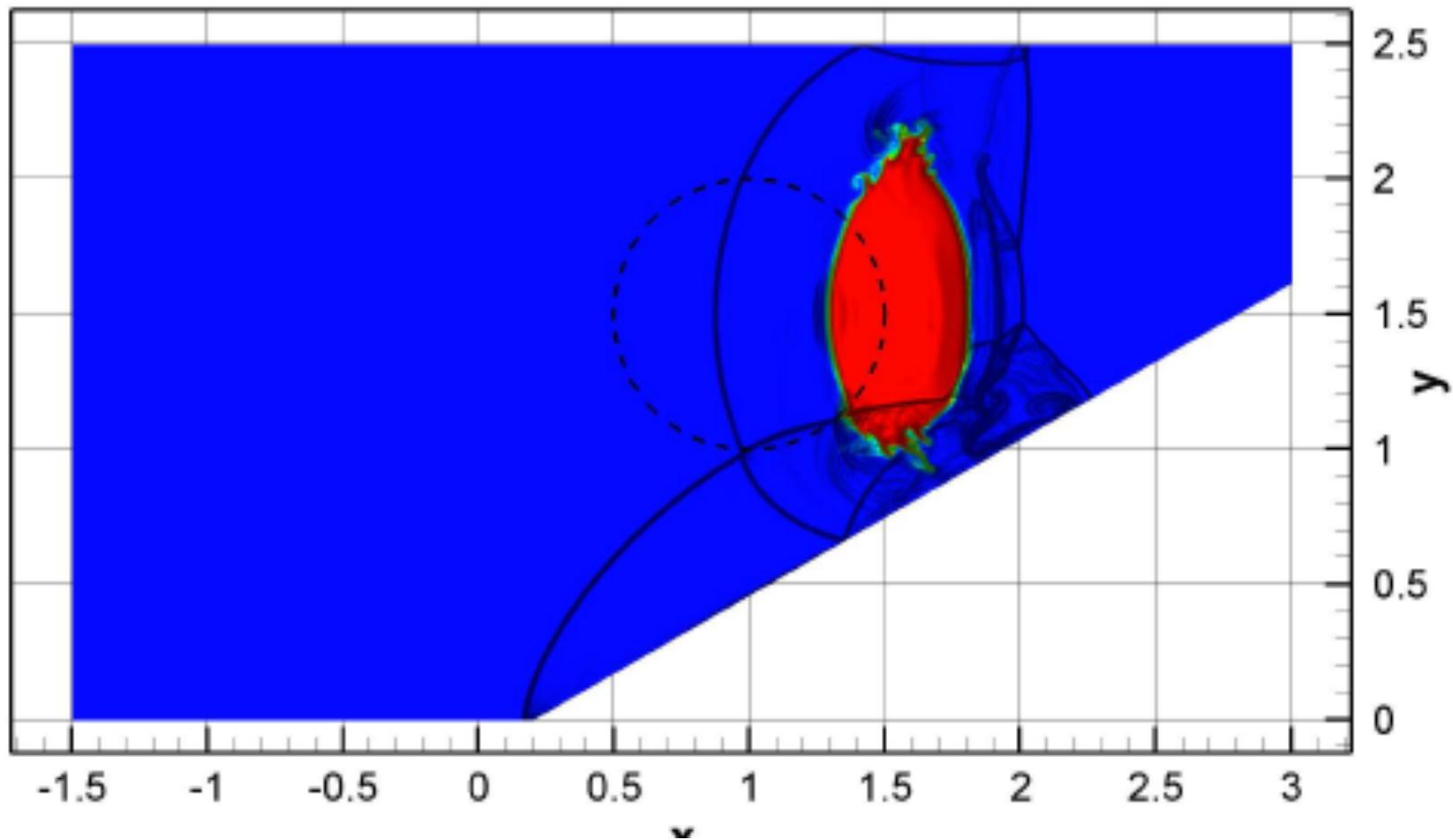
M Castro, A Pardo, C Pares and E F Toro  
*On some fast well-balanced first order solvers for  
nonconservative systems*

Mathematics of Computation. Vol. 79, pages 1427-1472, 2009.

# Double Mach reflection for the 2D Baer-Nunziato equations



## Double Mach reflection for the 2D Baer-Nunziato equations



## *Summary on FORCE*

- *A centred scheme*
- *One-step scheme*
- *In conservative form, with a numerical flux*
- *Monotone*
- *Linearly stable up to  $CFL = 1, 1/2, 1/3$*
- *Very simple to use, applicable to any system (useful for complicated systems)*
- *High-order extensions (TVD, WENO, DG, ADER)*
- Further reading: Chapter 18 of:

Toro E F. Riemann solvers and numerical methods for fluid dynamics.  
Springer, Third Edition, 2009.

*Thank You*