Summer School « Non-homogeneous fluid and flows », Prague, August 27-31, 2012

Nonlinear properties of internal gravity waves
Part I : homogeneous medium

Chantal STAQUET

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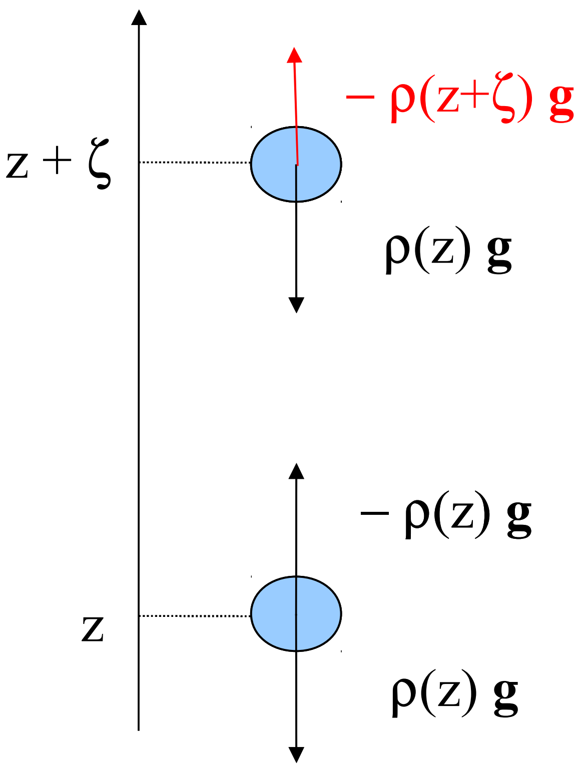
Internal gravity waves ?

Oscillations of a fluid particle in a stably stratified fluid



- Consider a fluid particle of unit volume in a stably-stratified medium, located at altitude z at equilibrium position.

- Move the particle adiabatically upwards to altitude $z+\zeta$. The particle is no longer in equilibrium, being subject to the *buoyancy force :*

$$-\rho(z+\zeta)\,\mathbf{g} + \rho(z)\,\mathbf{g} \approx -g\, d\rho/dz\, \zeta$$

- Apply Newton's law to the particle :

$$\rho(z)\, d^2\zeta/dt^2 = -\, g\, d\rho/dz\, \zeta \quad \text{or}$$

$$d^2\zeta/dt^2 + N^2 \zeta = 0$$
$$\text{with } N^2 = -\, g/\rho(z)\, d\rho/dz$$

N(z) is the Brunt-Vaisala (or buoyancy) frequency.

Internal gravity waves

- In a continuous medium, any perturbation with a vertical component gives rise to waves. The restoring force is the buoyancy force $-\vec{g}\frac{d\rho}{dz}\zeta$ where ζ is the vertical displacement associated with the perturbation.

- What is (and how to find) the dispersion relation of these waves ?

Consider a medium in which the function N(z) is constant (*homogeneous* medium).

Start from the Navier-Stokes equations in the Boussinesq equations.
Linearize these equations about a rest state. These equations can be reduced to a single equation for the vertical velocity component *w* :

$$\frac{\partial^2 \nabla^2 w}{\partial t^2} + N^2 \nabla_H^2 w = 0,$$

where $\nabla_H^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian.

Internal gravity waves (continued)

- Linearized Boussinesq equations about the rest state, *for N constant*, written for *w* :

$$\frac{\partial^2 \nabla^2 w}{\partial t^2} + N^2 \nabla_H^2 w = 0.$$

This is a PDE with constant coefficients, in an infinite medium.

- One can therefore look for a plane wave solution: $w(\vec{x}, t) = W e^{i(\vec{k}.\vec{x} - \omega t)}$

which requires that

$$\boxed{\omega^2 = N^2 \sin^2 \theta}$$

where θ is the angle that the group velocity makes with the vertical. This is the *dispersion relation* of internal gravity waves, valid for a homogeneous medium.

- In a rotating medium with angular velocity Ω, the dispersion relation becomes

$$\boxed{\omega^2 = N^2 \cos^2 \theta + (2\Omega)^2 \sin^2 \theta}$$

$$\rightarrow (2\Omega)^2 \leq \omega^2 \leq N^2$$

Properties of internal gravity waves : illustrations

Internal gravity waves emitted by an oscillating paddle :
http://www.phys.ocean.dal.ca/programs/doubdiff/pics/iw1.mpeg

Internal gravity waves emitted by an impulse :
http://www.phys.ocean.dal.ca/programs/doubdiff/pics/iw4.mpeg

Evidence of internal gravity waves

1. In laboratory experiments

The experiment of Görtler (1943)

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Inhalt:

HAUPTAUFSÄTZE

Über eine Schwingungserscheinung in Flüssigkeiten mit stabiler Dichteschichtung.

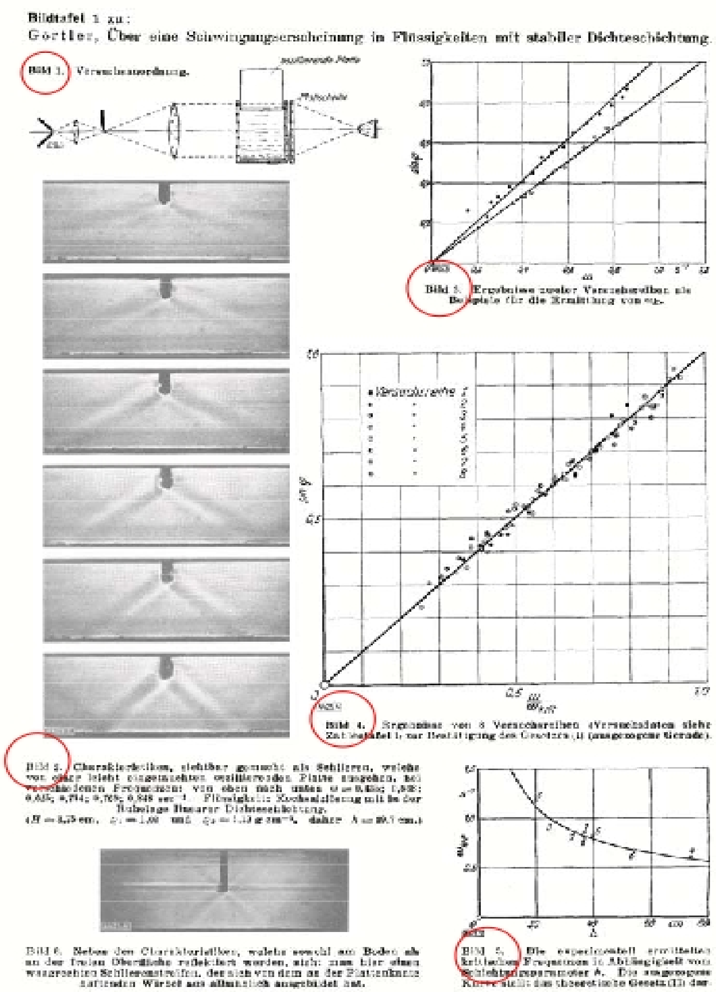
Von H. Görtler in Göttingen.
(Aus dem Kaiser-Wilhelm-Institut für Strömungsforschung.)

Die Theorie der erzwungenen kleinen Schwingungen in einer der Schwerkraft unterworfenen reibungslosen Flüssigkeit mit stabiler Dichteschichtung führt zur Voraussage einer neuartigen Schwingungserscheinung: Die Differentialgleichung zweiter Ordnung für die Verschiebungsamplituden zeitlich harmonischer Schwingungen als Funktionen der Ortskoordinaten ist für Frequenzen unterhalb eines von der jeweiligen Dichteschichtung abhängigen kritischen Werts vom hyperbolischen Typ. Demgemäß treten Regularitätsdefekte dieser Größen an einem Orte zugleich in allen Punkten der durch diesen Ort gehenden reellen Charakteristiken der Differentialgleichung auf. In einfachen Experimenten läßt sich diese Erscheinung beobachten. Die theoretischen Gesetzmäßigkeiten des Charakteristikenverlaufs im einzelnen erfahren dabei ihre Bestätigung.

§ 1. Einleitung.

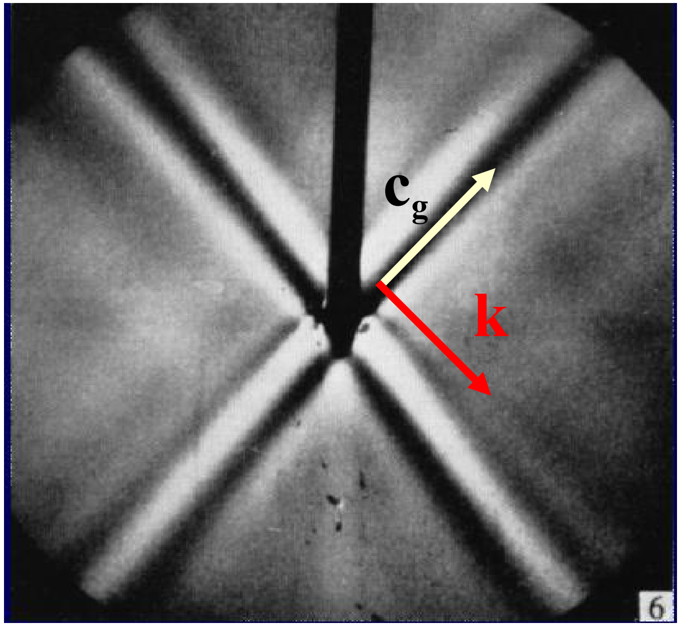
Kleine Schwingungen in reibungslosen, der Schwerkraft unterworfenen Flüssigkeiten oder Gasen mit in der Ruhelage horizontaler und stabiler Dichteschichtung sind mehrfach theoretisch untersucht worden[1]. Man interessierte sich vorwiegend für das Problem der Wellenausbreitung in solchen Medien mit teilweise festen oder freien Berandungen. Auf diese Untersuchungen soll hier nicht näher eingegangen werden. Die vorliegende Betrachtung gründet sich auf einen bemerkenswerten mathematischen Sachverhalt bei den Bewegungsgleichungen zeitlich harmonischer Schwingungen, dem bisher merkwürdigerweise nicht die verdiente Beachtung geschenkt worden ist. Dies überrascht um so mehr, als bereits durch sehr einfache und naheliegende Schlußfolgerungen aus diesem Sachverhalt theoretische Vor-

[1] Genannt seien: Lord Rayleigh: Investigation on the character of the equilibrium of an incompressible heavy liquid of variable density. Proc. Lond. Math. Soc. Bd. 14 (1883), S. 170; W. Burnside: On the small wave-motions of a heterogeneous fluid under gravity. Proc. Lond. Math. Soc. Bd. 20 (1889), S. 392; A. E. H. Love: Wave motion in a heterogeneous heavy liquid. Proc. Lond. Math. Soc. Bd. 22 (1891), S. 307; H. Lamb: On the theory of waves propagated vertically in the atmosphere. Proc. Lond. Math. Soc. (2) Bd. 7 (1909), S. 122; On atmospheric oscillations. Proc. Roy. Soc. Bd. 84 (1911), S. 551; S. Sano: Some problems on waves propagating in air at an uniform temperature. Bull. Central Meteorol. Observatory, Japan. Bd. 2 (1913), S. 17. Siehe ferner: V. Bjerknes, J. Bjerknes, H. Solberg und T. Bergeron: Physikalische Hydrodynamik. Berlin 1933 (achtes Kapitel) und H. Lamb: Lehrbuch der Hydrodynamik. Deutsch von E. Helly. 2. Aufl., Leipzig und Berlin 1931, S. 603 bis 626.
Den Hinweis auf die Arbeit von S. Sano verdanke ich Herrn H. Ertel.



Evidence of internal gravity waves
1. In laboratory experiments (cont'd)

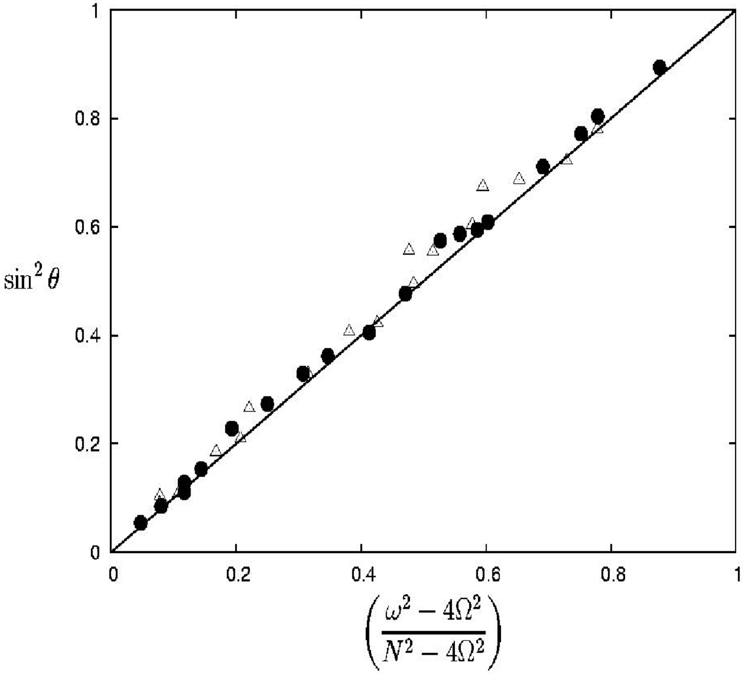
Non rotating medium
The Mowbray and Rarity (1967) experiment



Dispersion relation

$$\omega^2 = N^2 \sin^2\theta$$

Rotating medium
(from Peacok and Tabaei, 2005)



Dispersion relation

$$\omega^2 = N^2 \sin^2\theta + (2\Omega)^2 \cos^2\theta$$

Evidence of internal gravity waves
2. In the atmospheric boundary layer

SOLITONS AND OTHER STRUCTURES IN THE STABLE BOUNDARY LAYER

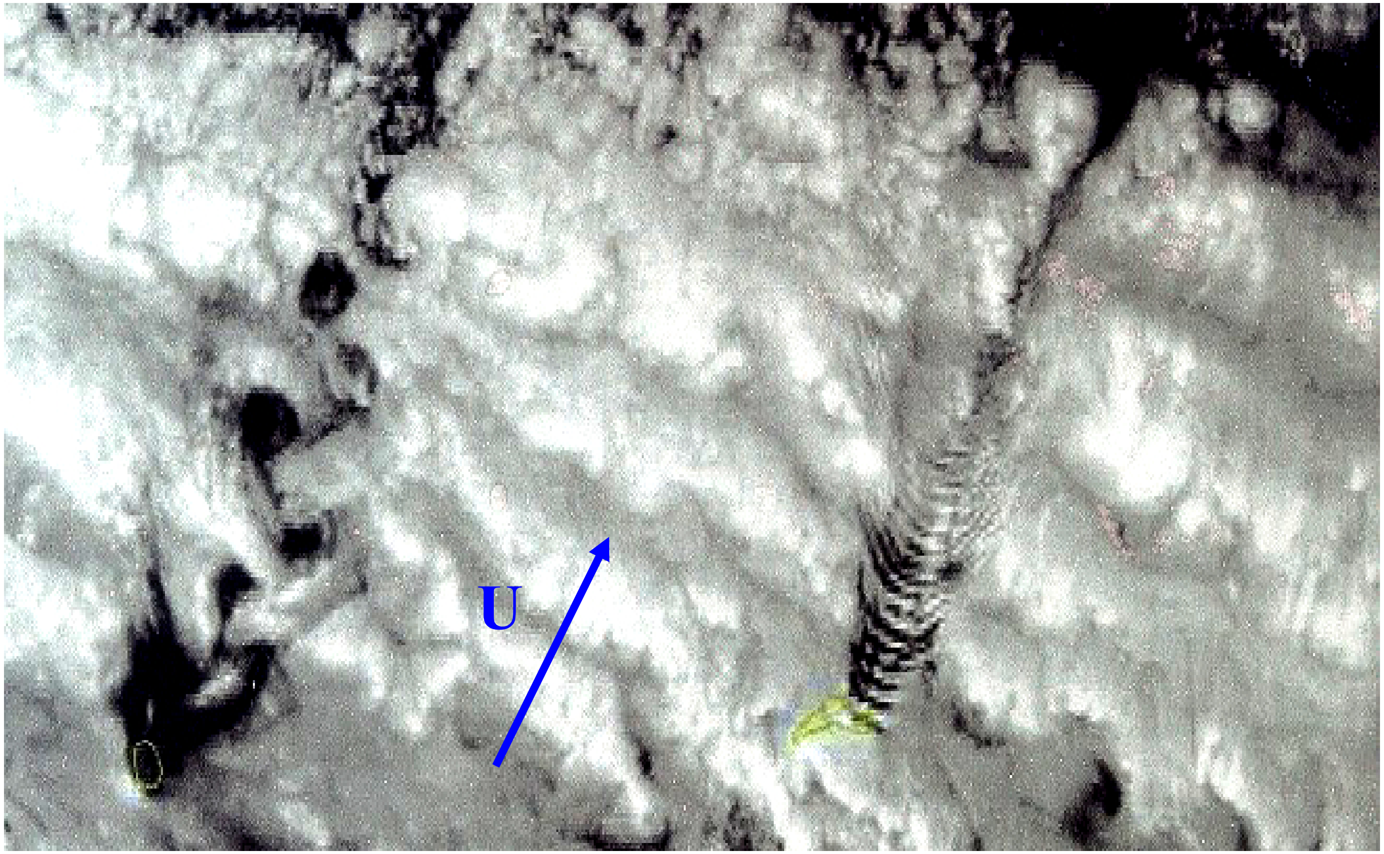
Larry Mahrt and Bob Heald

This video shows a wide variety of structures in the stable nocturnal boundary layer taken at dawn. The video is speed up generally by a factor of 32 x to better visualize the motions.

http://youtu.be/eydofsyX3Ns

Evidence of internal gravity waves
3. In the atmosphere

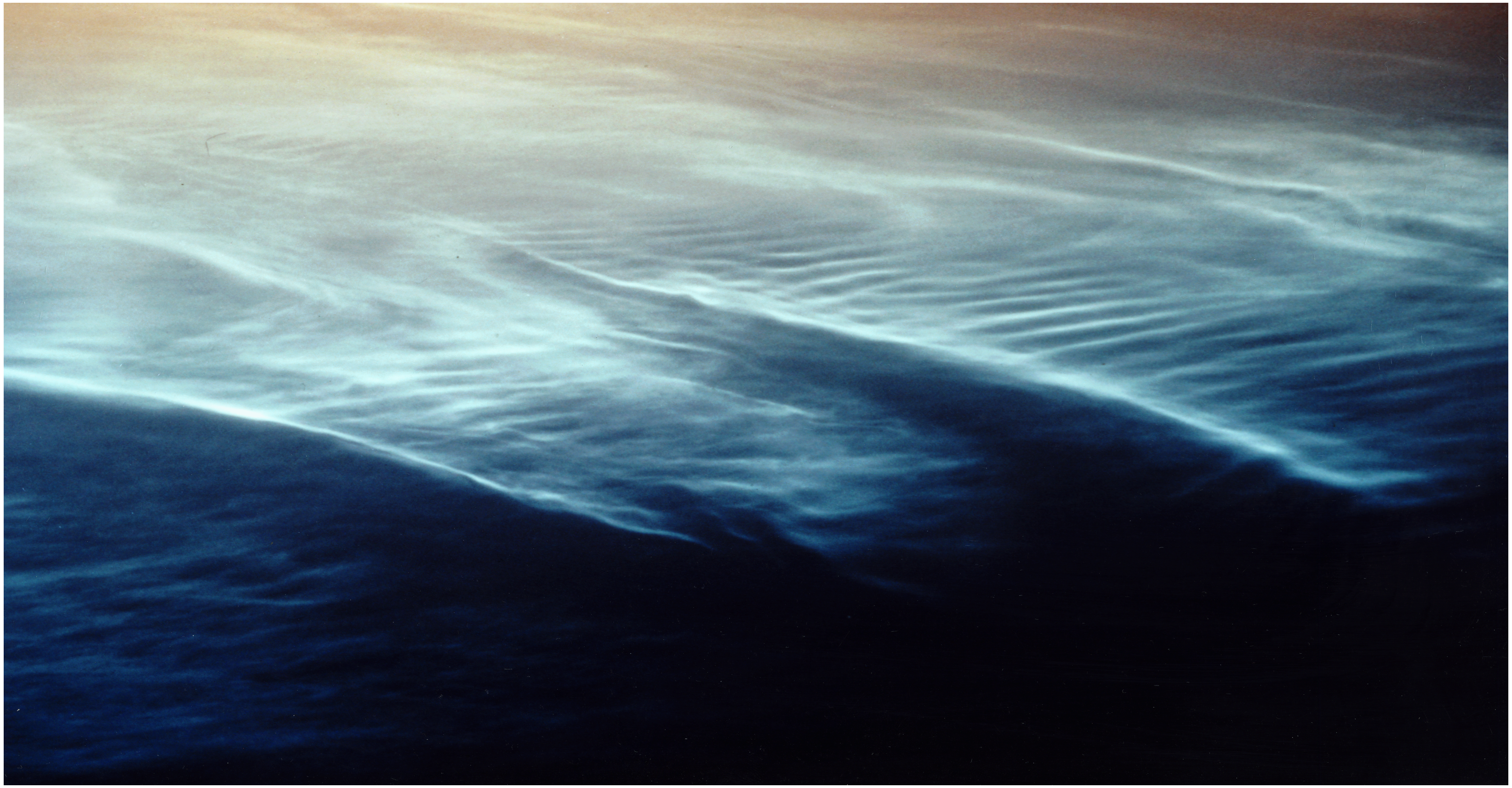
Blowing of the wind over two islands ...



A 'high elevation' island :
von Karmann street
U/(NH) < 1

A 'small elevation' island :
Internal gravity waves (« lee waves »)
U/(NH) > 1

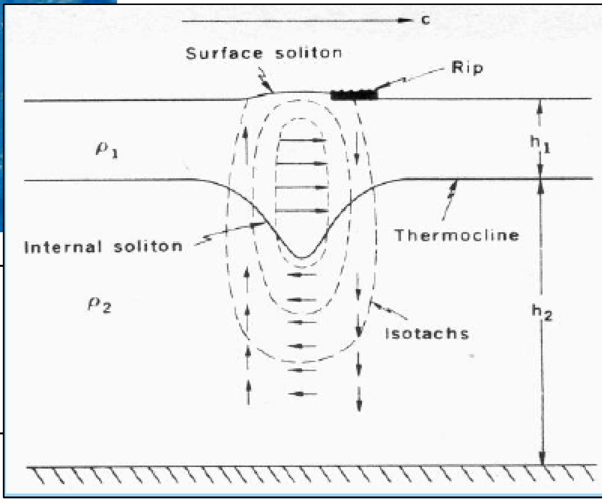
Evidence of internal gravity waves
4. In the mesosphere



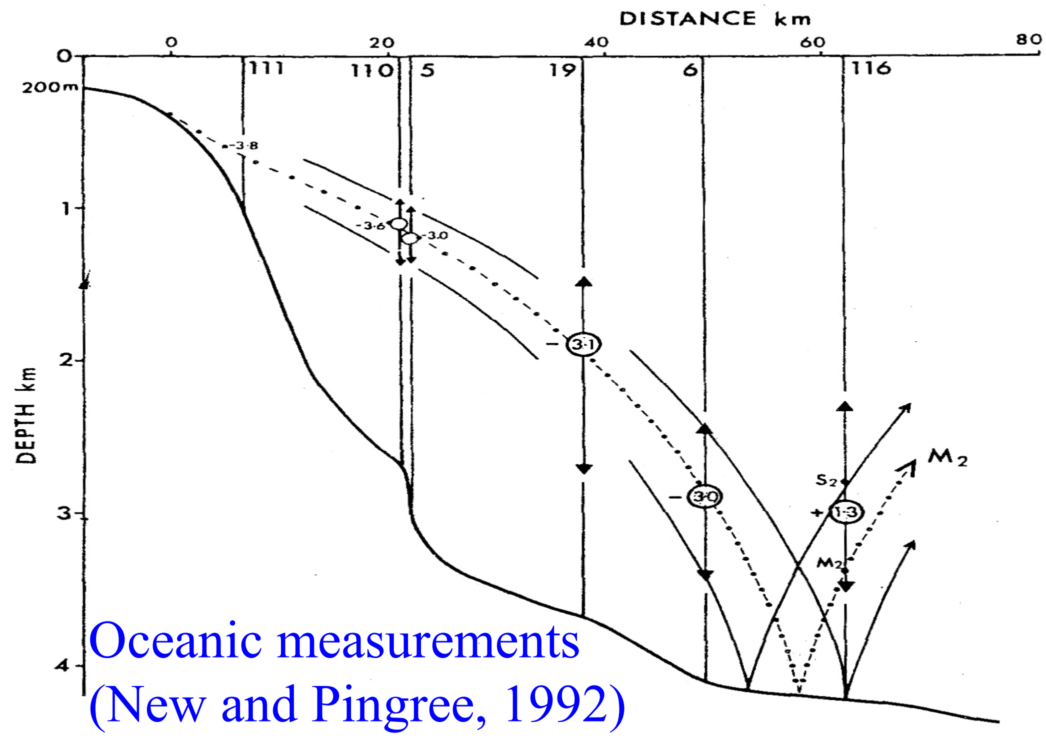
View of noctilucent clouds (in the mesosphere, at 80 km altitude) from Kustavi, Finland (61° N, 21° E) on 22 July 1989 showing phase planes and streak structures. The planes are separated by about 50 km and streaks by 3 to 5 km (from Fritts et al. 1993; photograph by Pakka Parviainen).

Evidence of internal gravity waves
5. In the upper ocean

Internal waves (propagating *below* the free surface) generated by the interaction of the tide with a continental slope (SAR image).

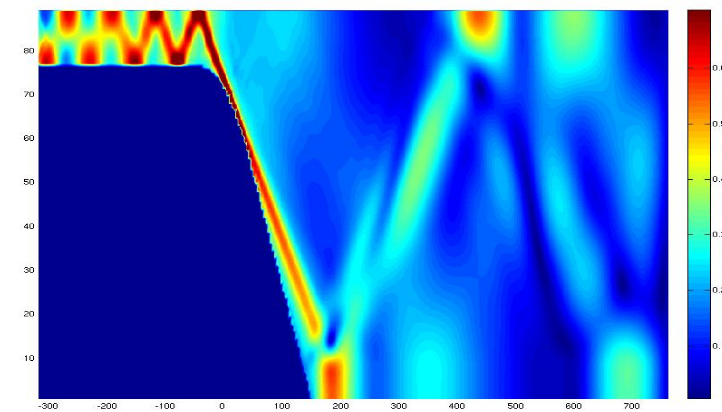


Evidence of internal gravity waves
6. The internal tide



Oceanic measurements
(New and Pingree, 1992)

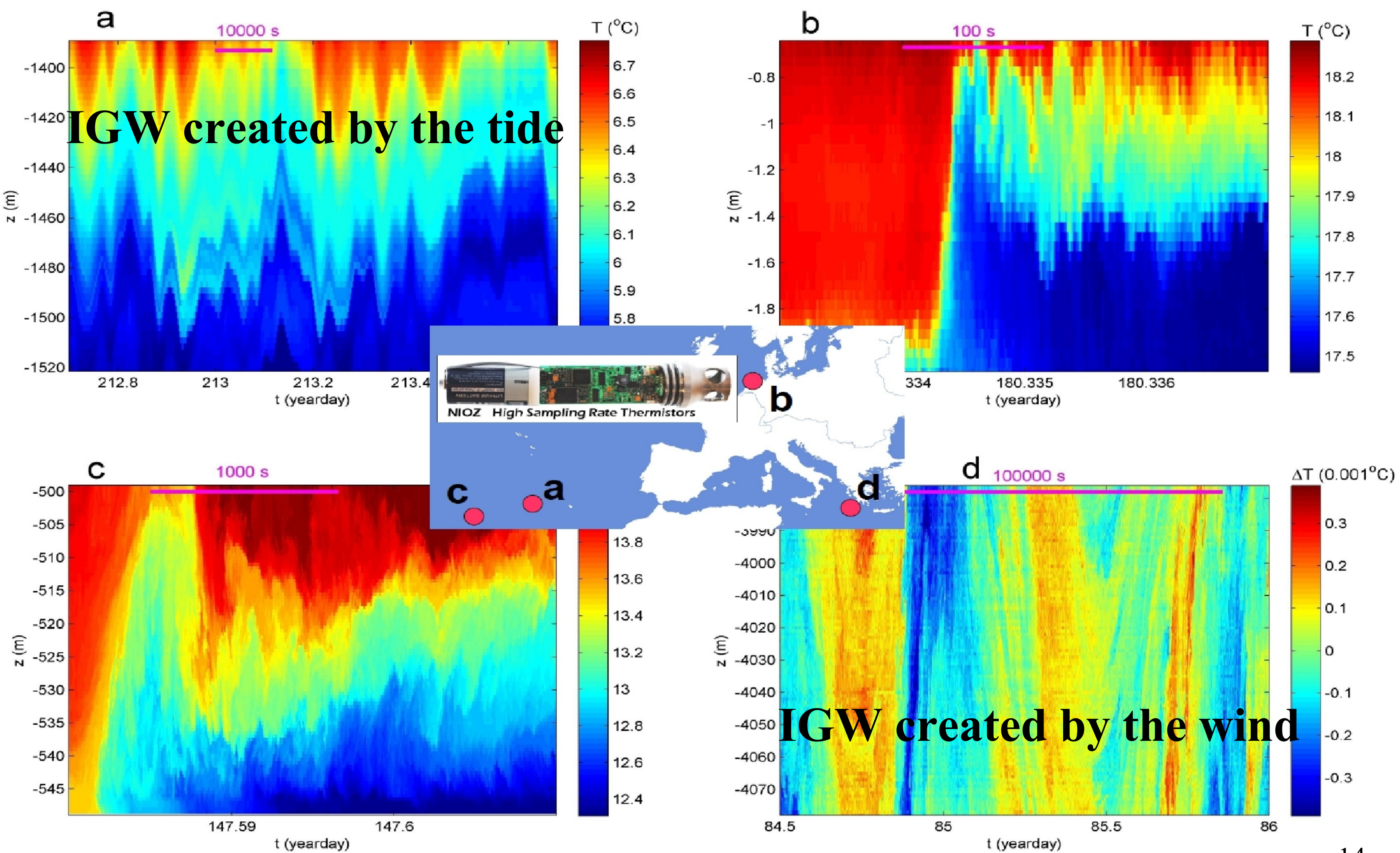
FIG. 9. Diagram showing the theoretical ray path (chained line) for a beam of internal tidal energy at the M_2 tidal frequency emanating from the critical depth (385 m) on the upper slopes and reflecting off the Biscay abyssal plain at a depth of about 4200 m, 58 km from the critical point. Also shown is a summary of the internal tidal oscillations obtained during the RRS *Challenger* cruises in 1988 (CH 31/88) and 1987 (CH 18/87). Vertical lines represent mooring and CTD station positions and are identified with numbers. CTD stations 5 and 6 and mooring 116 are from the 1988 cruise, whereas moorings 110 and 111 and CTD 19 were obtained in 1987. The depth of the maximum amplitude of the internal tidal oscillation found at each station is plotted as an open circle and the range where the amplitude is more than 70% of the maximum value is indicated by the arrows. Two further rays are shown (solid lines) passing through the 70% limits near mooring 110. The phase of the maximum upward displacement is given (within the circles) in hours with respect to HWP. A ray at the M_2 tidal frequency would intersect mooring 116 at the depth marked M_2; S_2 is the corresponding point for a ray at the S_2 tidal frequency. The topography is depicted by the bold line and is critical at 385 m: the horizontal distance scale is measured from the critical point.



Numerical simulations (N cst)
(Pairaud et al. 2010)

Evidence of internal gravity waves

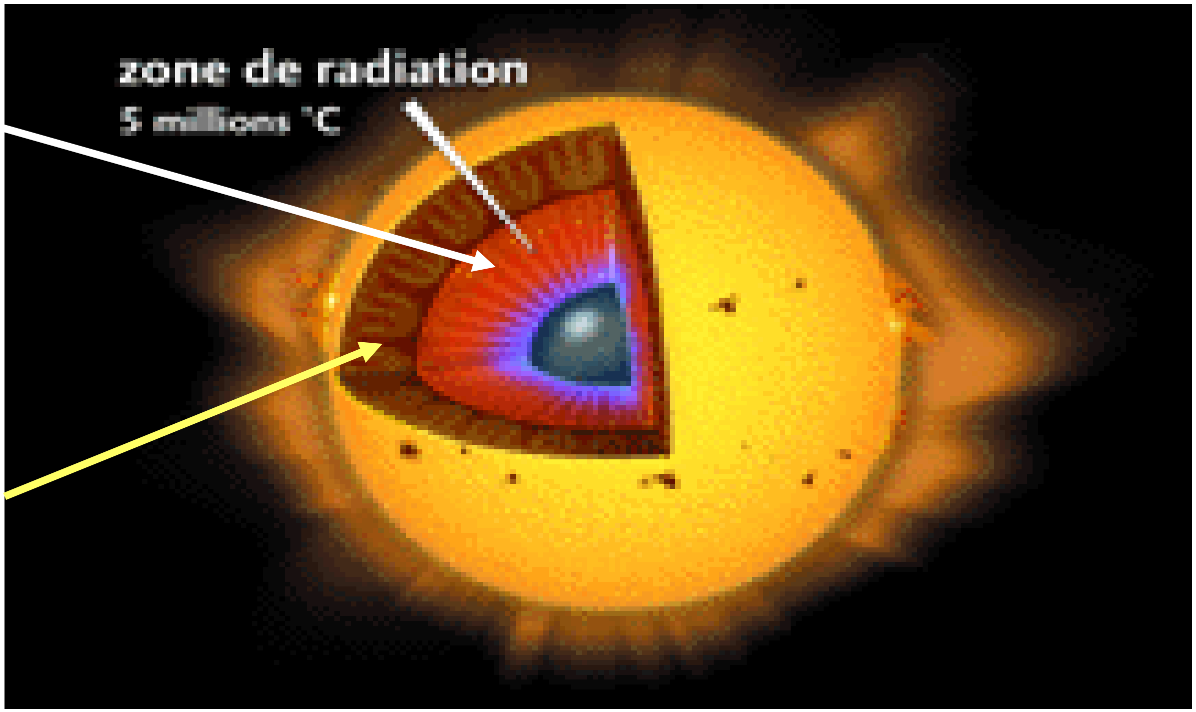
7. In the deep ocean



Evidence of internal gravity waves
8. In the interior of the Sun

Radiative zone

Convective zone



Transport of Lithium by (weakly nonlinear) IGW toward the core in the radiative zone → burning of Li → account for under-abundance of Li in the Sun ? (Schatzman, JFM, 1996).

Generation of internal gravity waves in the Atmosphere and the Ocean (brief summary)

- In the atmosphere the main generation process is the wind blowing over topography → « lee waves ».

- In the ocean, the main generation processes are :
  - the wind blowing at the surface of the ocean (→ near-inertial waves);
  - the tide passing over topography → « internal tide » ;
  - in the Southern Ocean : the Antarctic Circumpolar Current passing over bottom topography → « lee waves ».

- Most studies of internal gravity waves address the instability, breaking and mixing properties of internal gravity waves (i.e. their nonlinear properties).

Why ?

Generation of internal gravity waves in the Atmosphere and the Ocean
(brief summary)

- In the atmosphere the main generation process is the wind blowing over topography → « lee waves ».

- In the ocean, the main generation processes are :
  - the wind blowing at the surface of the ocean ;
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- Most current studies of internal gravity waves address the instability, breaking and mixing properties of internal gravity waves (i.e. their nonlinear properties).

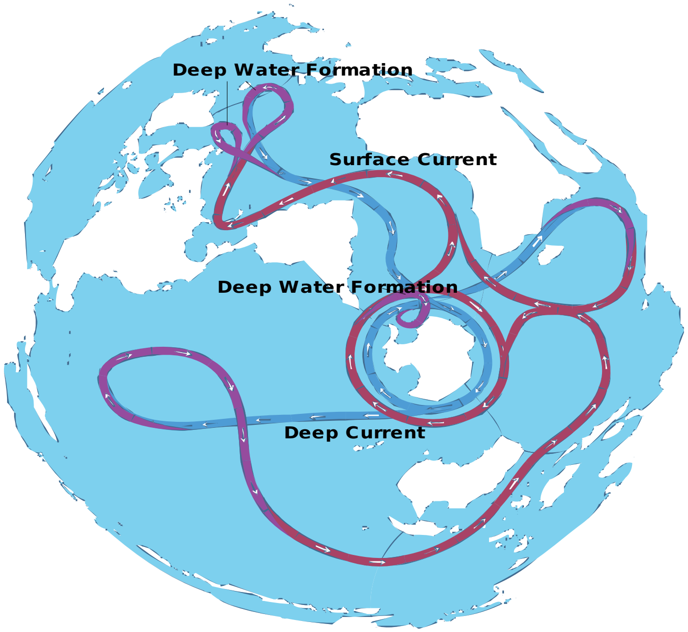
Why ?

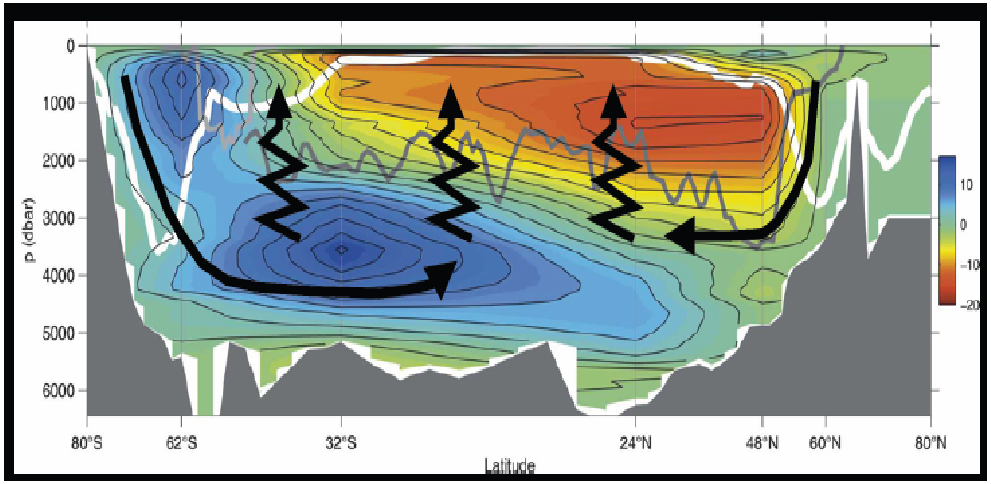
Linear non-dissipative internal gravity waves transport the momentum and the energy of the source which emits them **without transport of mass** (non-acceleration theorem).

→ Nonlinear (finite amplitude or breaking) dissipative waves induce transport properties :

- through momentum deposition in the atmosphere
  → strong impact on ambient winds
- through mixing of mass and temperature in the ocean
  → strong impact on thermohaline circulation.

The thermohaline circulation





Zonally averaged stream function, expressed in Sv
(Lumpkin & Speer 2007)

Deep cold water masses of the thermohaline circulation return to the surface while mixing.

Mixing is supposed to occur through breaking of internal gravity waves.

OUTLINE

Lectures 1 and 2

- Parametric instability and breaking of a monochromatic internal gravity (IGW)
- Application to the oceanic internal tide
- Statistical properties of nonlinear IGW

Lectures 3 and 4

- Propagation of internal gravity waves in a non homogeneous fluid (in a background N(z) stratification)
- Propagation of internal gravity waves in a non homogeneous flow (in a background U(y,z) velocity field)
- Wave-induced mean flow

Parametric instability of a monochromatic internal gravity wave

Instability of a monochromatic internal gravity wave

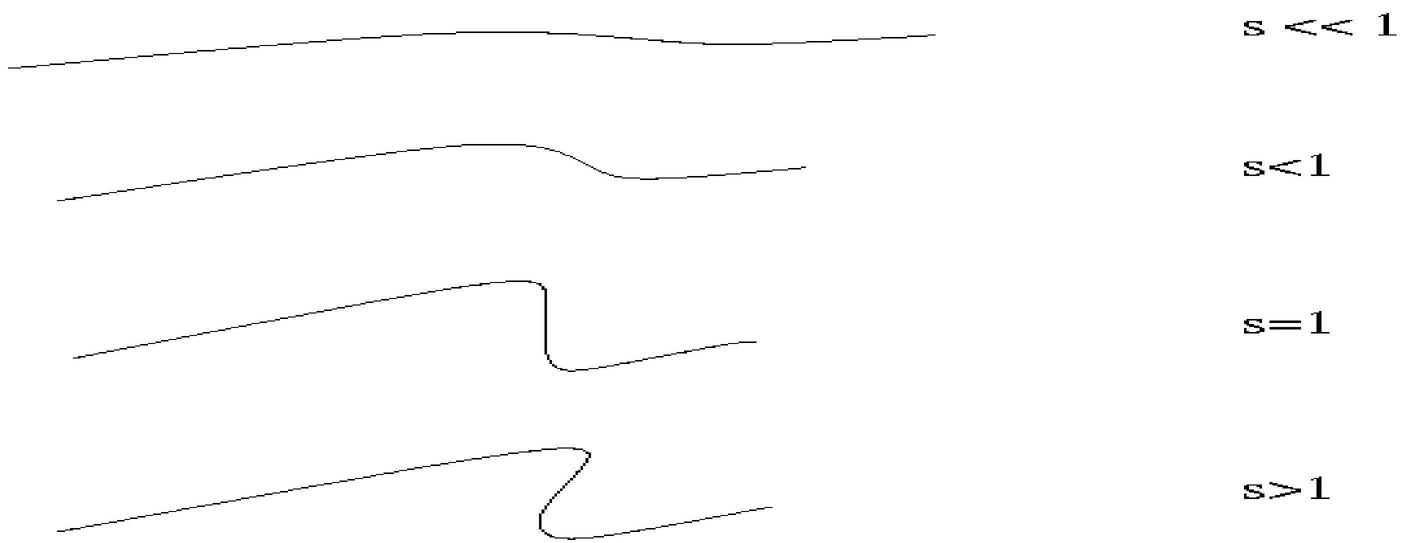
Definition of the wave steepness

In the following, we consider a monochromatic internal gravity wave ($\mathbf{k}_0$, Ω_0) and assume that fluid motions occur in the propagation plane ($\mathbf{k}_0$,$\mathbf{g}$) of the wave. This a valid assumption if the amplitude of the wave is infinitely small.

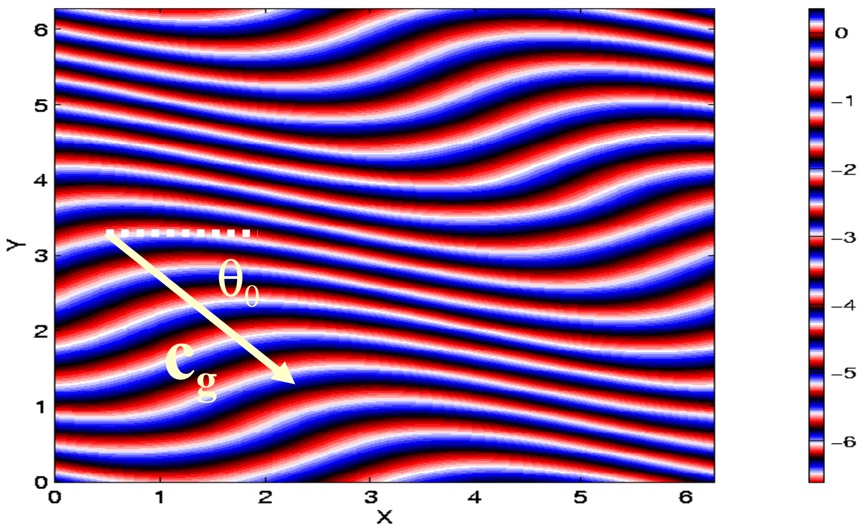
The normalized amplitude or steepness of the wave is defined as:

$$s = u_{max} / c_x$$

where c_x is the phase velocity in the *x*-direction.



Instability of a monochromatic internal gravity wave
Resonant interactions and parametric instability



Density contours of a **k**=(1,1) propagating gravity wave, with frequency N$\sin\theta_0$ and steepness s=0.25

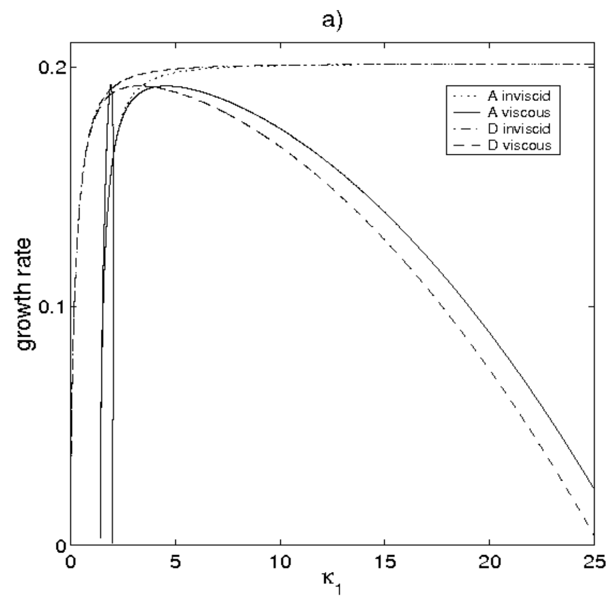
Any monochromatic internal gravity wave of steepness < 1 is unstable to parametric instability, whatever the statification level of the fluid (Drazin 1977).

This result can be derived from resonant interaction theory for s « 1 (Hasselmann 1967) and from stability analysis of the primary wave for s<1 (Mied 1976).

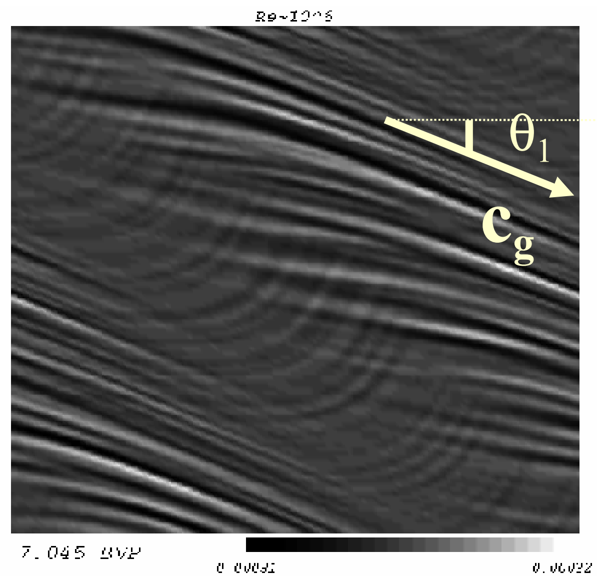
How does the parametric instability manifest itself ?

Instability of a monochromatic internal gravity wave
Resonant interactions and parametric instability

- The instability occurs in the propagation plane ($\mathbf{k_0},\mathbf{g}$) of the primary wave : it is a two-dimensional instability.
- The instability is selected by molecular effects and, in a numerical simulation, by the domain size as well.
- The instability manifests itself as "bands" with thickness equal to the molecular- selected scale and with angle equal to θ_1 / $N\cos\theta_1 = 0.5\, N\cos\theta_0$



Growth rate of the instability as a function of the wave vector modulus of the secondary wave.



Vorticity field of the secondary waves at t=7 ($2\pi/N$)

Koudella & Staquet JFM 2006

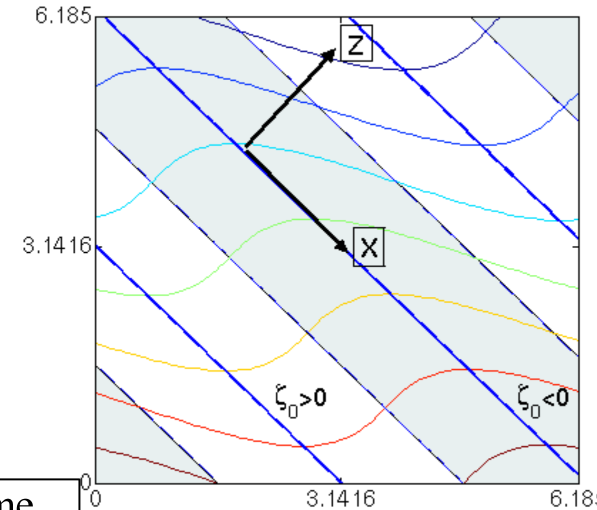
Parametric instability of a monochromatic internal gravity wave : an optimum scenario to perturbation growth

Dynamics of the primary wave best analysed in a rotated frame of reference (O,X,Z) with Z-axis // $\mathbf{k}_0$.

Let $[U_0,R_0]$ refer to the velocity and density fields of the primary wave and $[(u',w'),\rho']$ refer to those of the perturbation.

In this reference frame, the energy exchange between the perturbation and the primary wave occurs through two terms :

- **$- u'w' \, dU_0/dZ$** : kinetic energy exchange
- **$- \rho'w' \, dR_0/dZ$** : potential energy exchange (always > 0)



Let us assume u'w' is positive. Since dU_0/dZ oscillates in space and time, how is a non zero net forcing possible?

- If u'w' is around maximum when $dU_0/dZ < 0$, the KE perturbation is forced.
- If u'w' is around minimum (≈ 0) when $dU_0/dZ > 0$, little (no) KE energy is given back to the primary wave.

→ perturbation energy should be under **kinetic** form during the half primary wave period when **$dU_0/dZ<0$** and under **potential** form during the other half-period when **$dU_0/dZ>0$.**

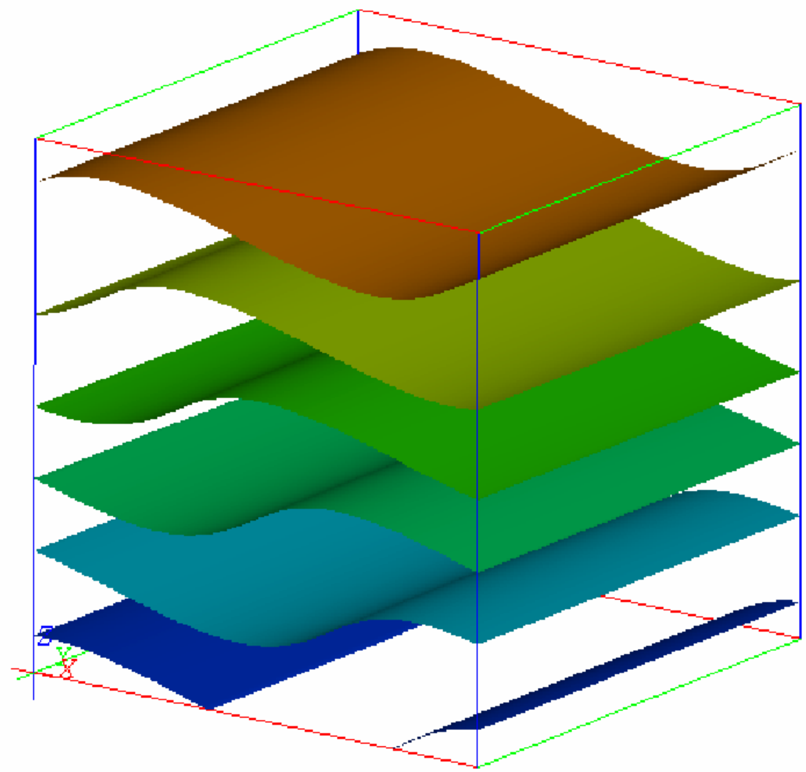
Hence the **perturbation kinetic energy** should have the same frequency as the **primary wave vorticity**, implying that the instability should be of **the parametric type.**

An approximate expression of these energy exchange terms can be computed from a kinematic model of the perturbation. In particular, we show that (Koudella & Staquet JFM 2006)

$\langle -u'w' \, dU_0/dZ\rangle / \langle -\rho'w' dR_0/dZ\rangle = \cos(\theta_1-\theta_0)$ (with $N\cos(\theta_1)=N\cos(\theta_0)/2$)

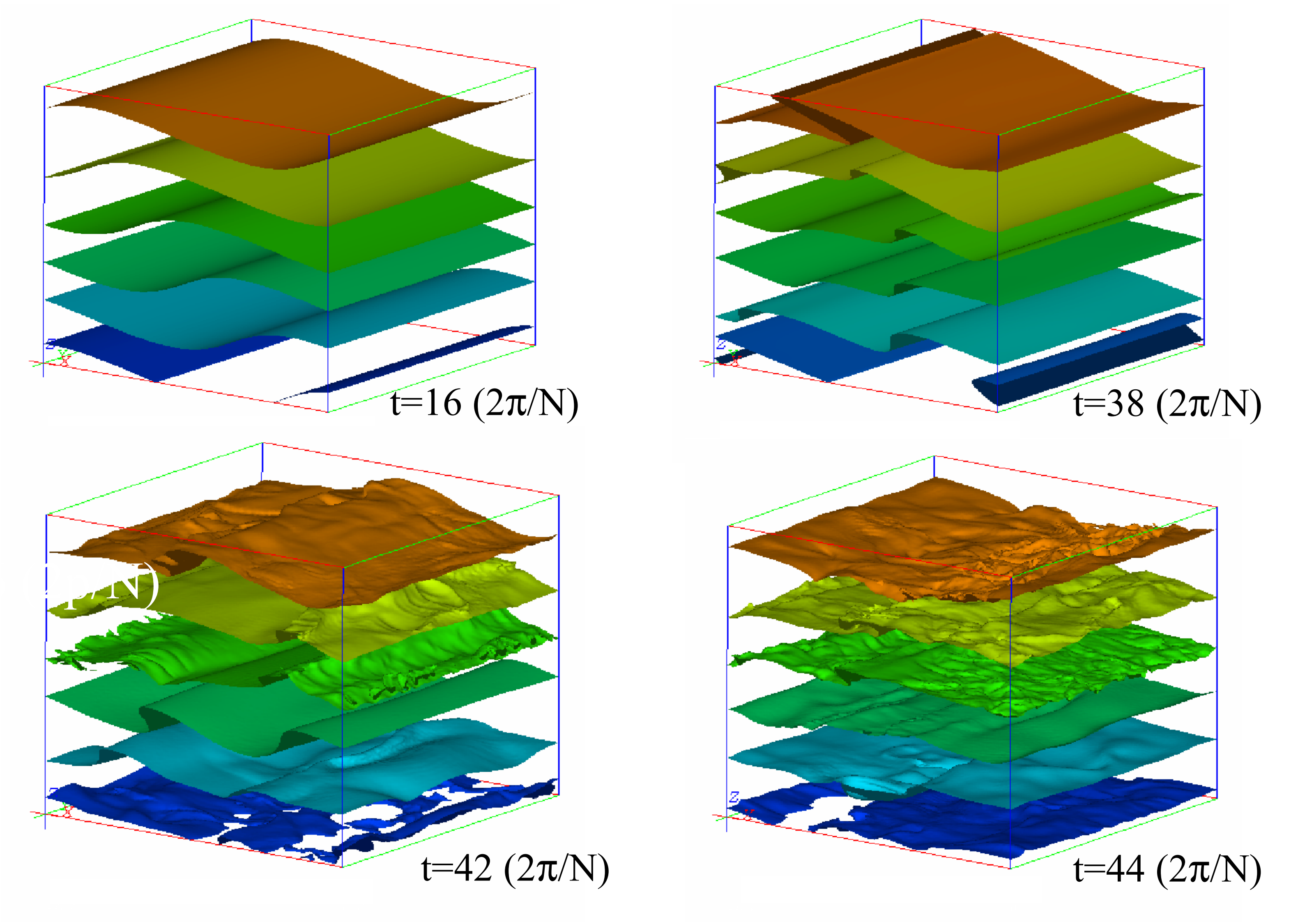
From parametric instability to breaking (for a monochromatic internal gravity wave).

Three-dimensional destabilization of a monochromatic internal gravity wave (s=0.3)



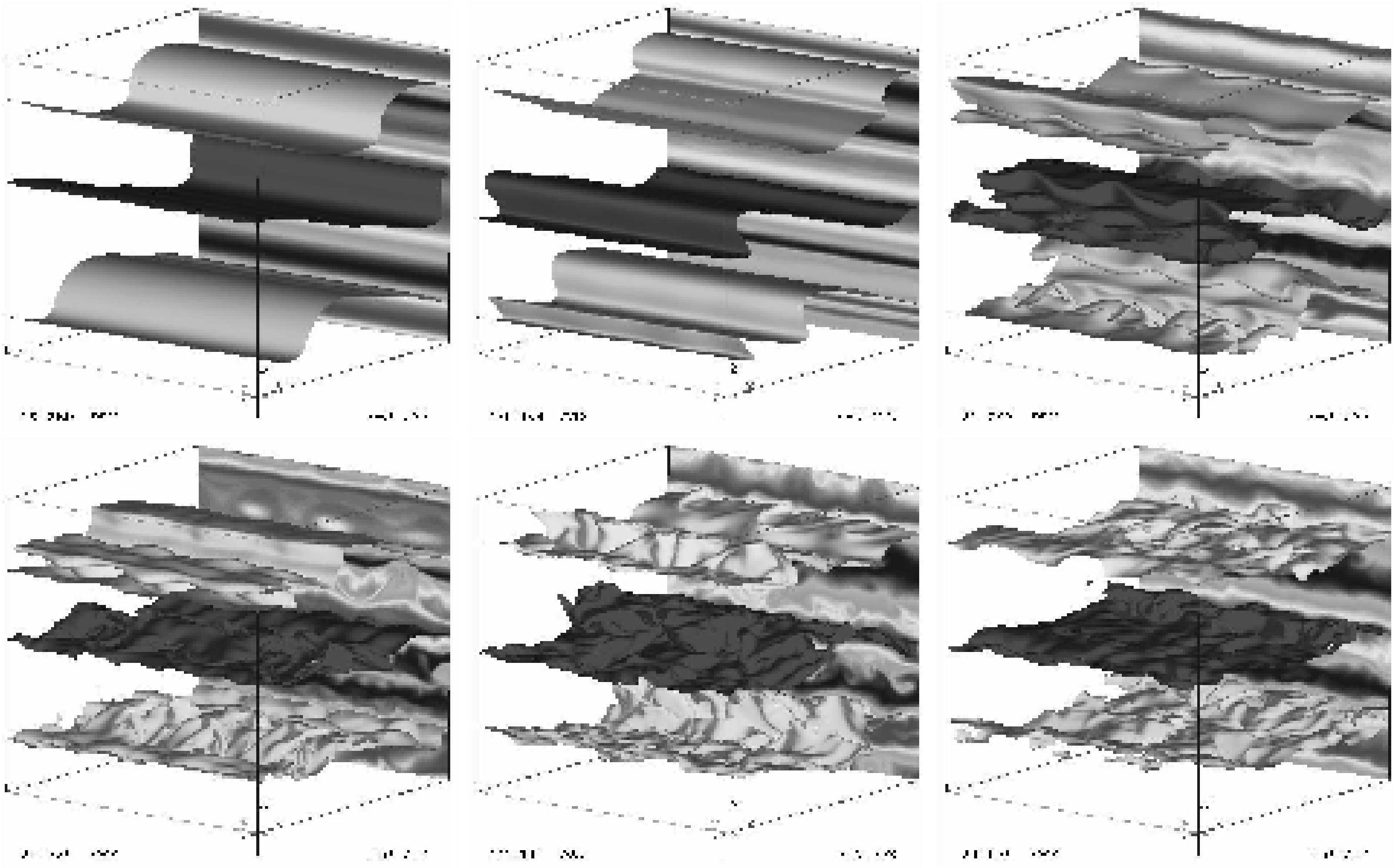
Six density surfaces for an unstable primary wave of initial steepness 0.3, plotted at t=16 ($2\pi/N$).

Three-dimensional destabilization of a monochromatic internal gravity wave (s=0.3)



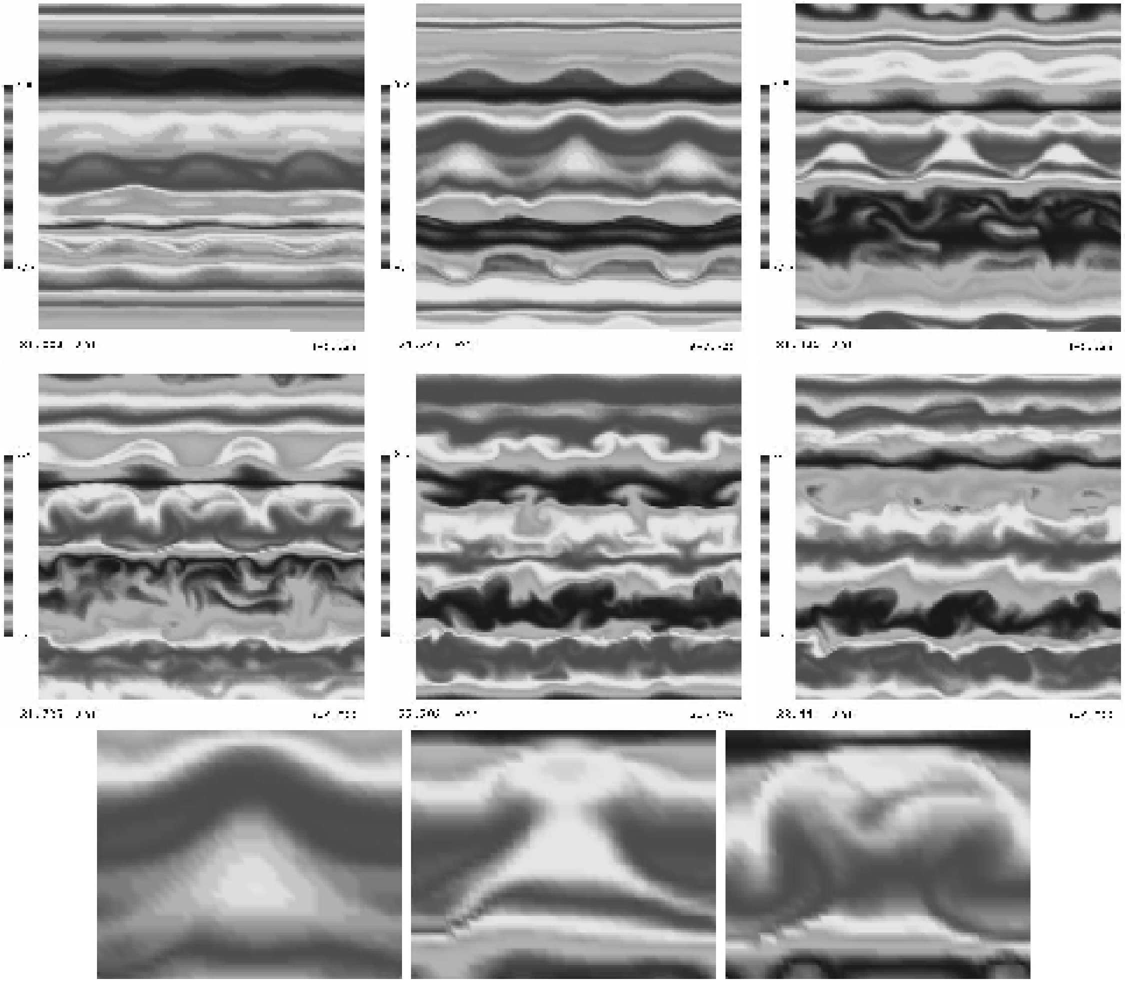
Six density surfaces at successive times.

Three-dimensional destabilization of a monochromatic internal gravity wave (s=0.7)



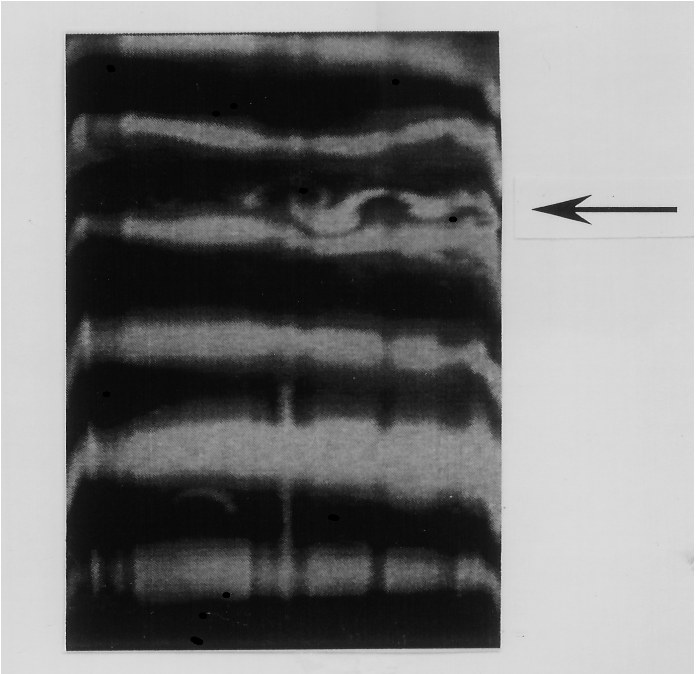
Hence, the amplification of parametric instability leads to locally overturned isopycnals ...

and the flow breaks through a buoyancy-induced instability ...

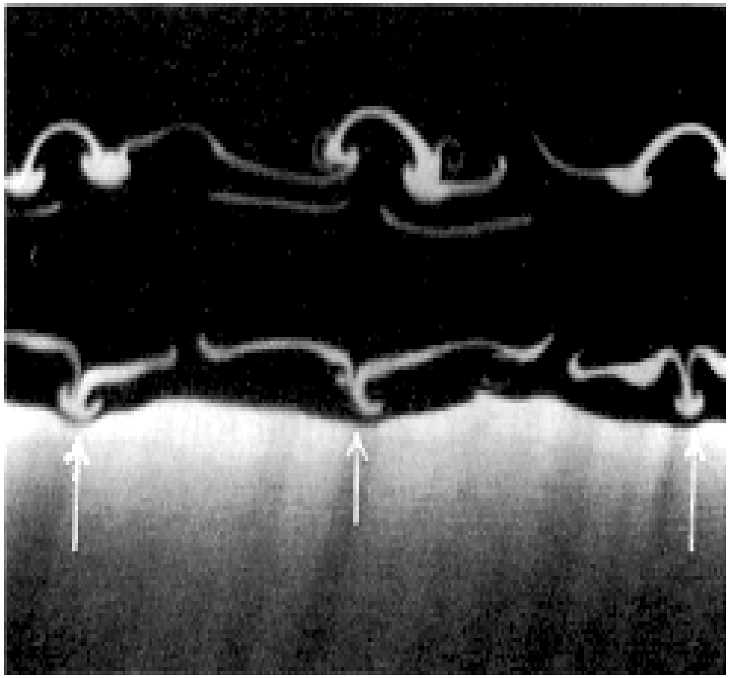


Cross-section of the density field in a plane perpendicular to the propagation plane of the primary wave, at successive times.
The development of a buoyancy-induced (or Rayleigh-Taylor) instability is displayed.
The initial steepness of the primary wave is 0.7.

... as in laboratory experiments

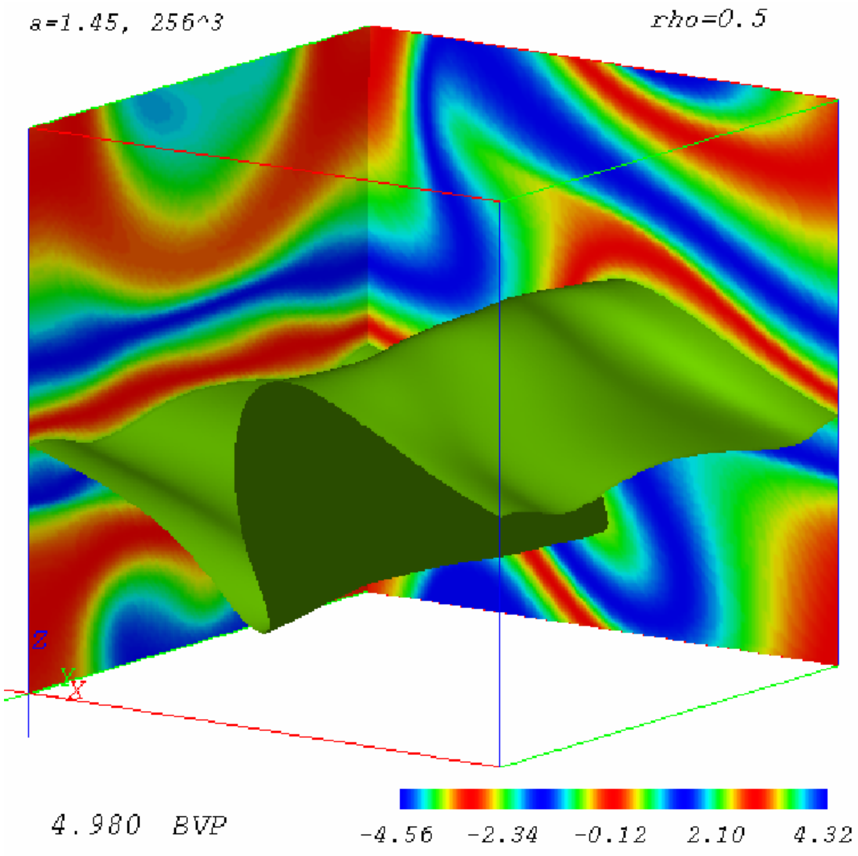


Breaking internal gravity wave (Benielli & Sommeria 1998)



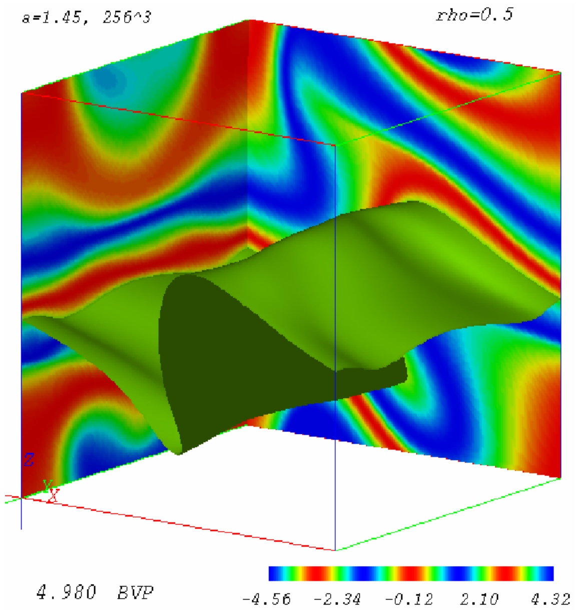
Stably-stratified shear layer (Schowalter, Lasheras & van Atta 1994)

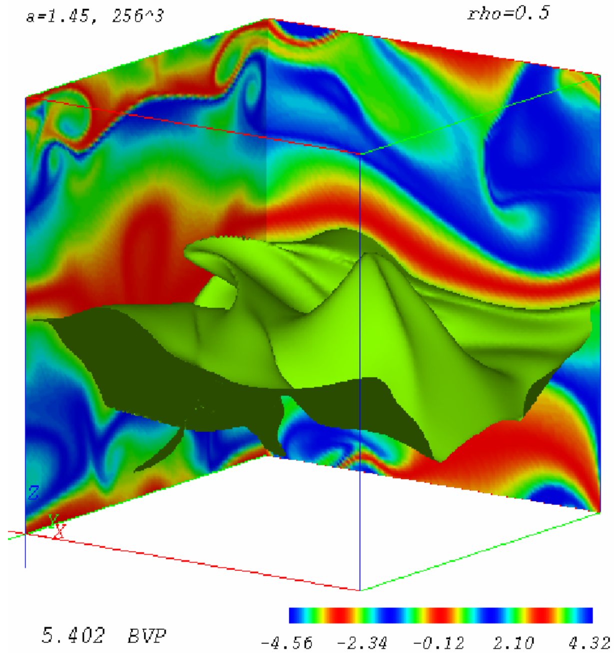
What does happen if the initial steepness is greater than 1?

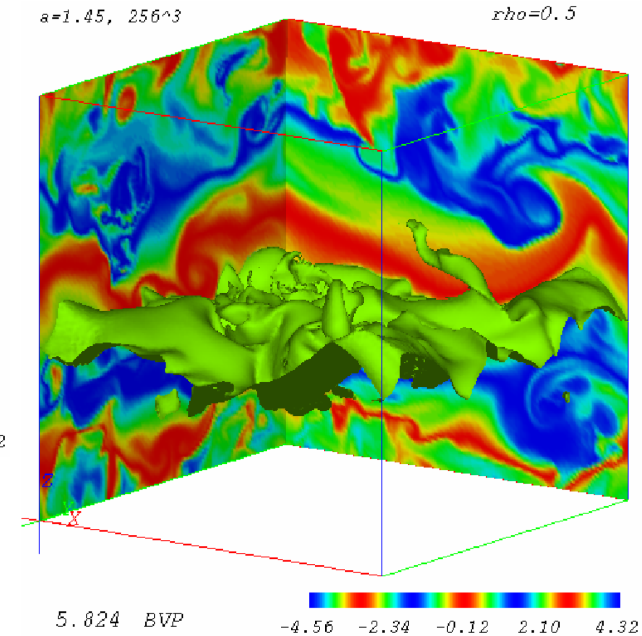


Density surface of a s=1.45 monochromatic internal gravity wave at 5 (2π/N)
(i.e. close to initial time)

The wave breaks through a buoyancy-induced instability







Density surface of a s=1.45 monochromatic internal gravity wave at 5(2π/N), 5.4(2π/N) and 5.8(2π/N)

Evidence for parametric instability in geophysical fluids ?

In the atmosphere

The wave amplitude grows as the wave propagates upwards so that parametric instability (with growth rate ~ 1/s) should not commonly occur.

In the ocean

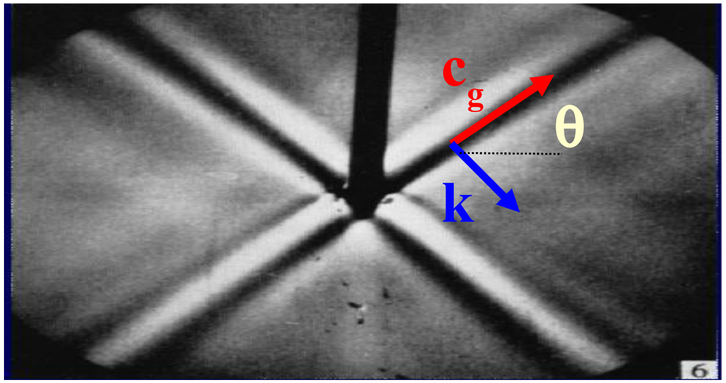
The instability may have time to develop in the ocean because ocean is an incompressible (and confined) medium.

→ the primary wave field should be sustained during the slow growth of the instability (growth rate ~ 1/s). This is the case for the internal tide.

Parametric instability (PSI) of an internal tide beam and mixing in the ocean

1. PSI in internal tide from joint laboratory experiments and numerical simulations (no rotation)
2. The influence of rotation on PSI in internal tide from numerical simulations at oceanic scale
3. Evidence of PSI from field campaigns of internal tide
4. The possible rôle of PSI in oceanic mixing

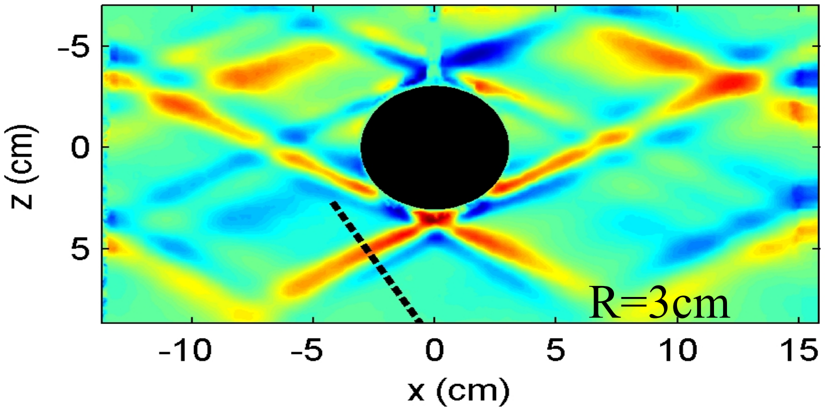
Emission of internal gravity waves by an oscillating body
(→ generation of beams)



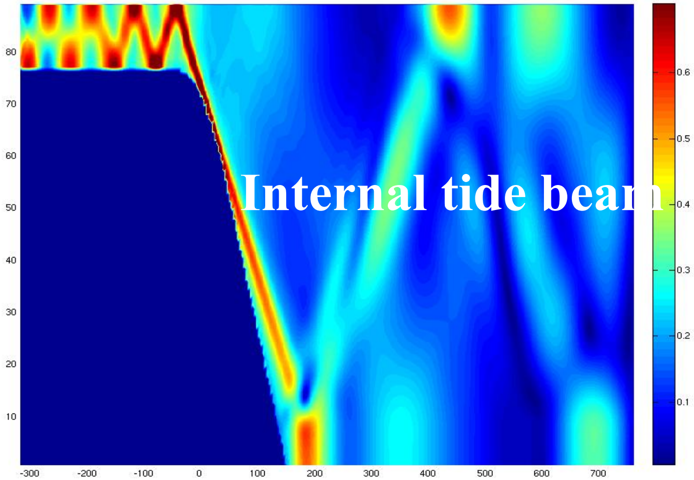
Mowbray & Rarity 1967 (f=0)

$$\omega^2 = N^2 \sin^2\theta$$

$(+\ f^2 \cos^2\theta$, where $f = 2\Omega \sin\Phi$ is the Coriolis parameter)

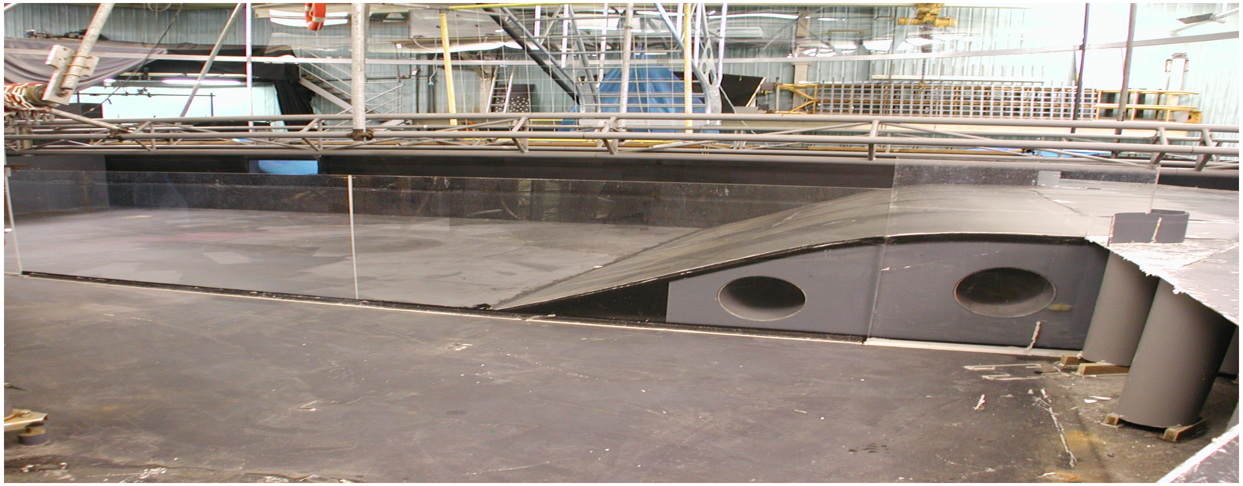


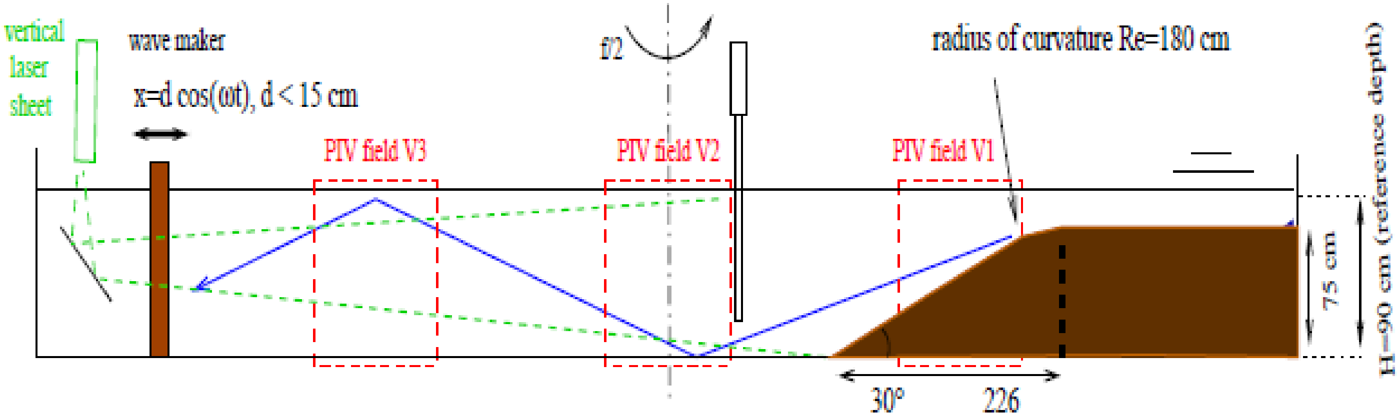
Gostiaux 2006



Pairaud et al. 2010

Laboratory experiments on the Coriolis platform (LEGI, Grenoble)

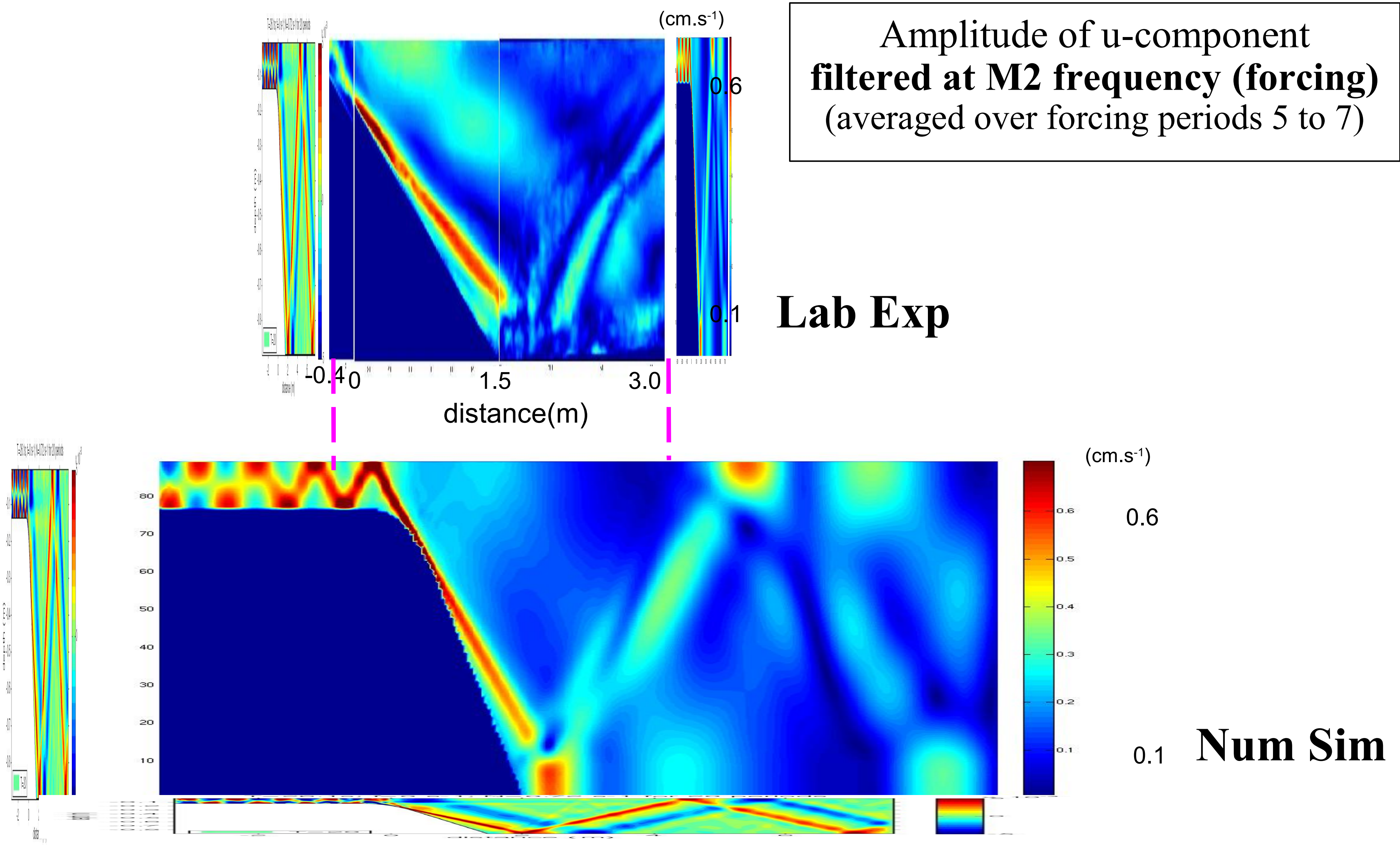




vertical cut

The internal tide field at early times

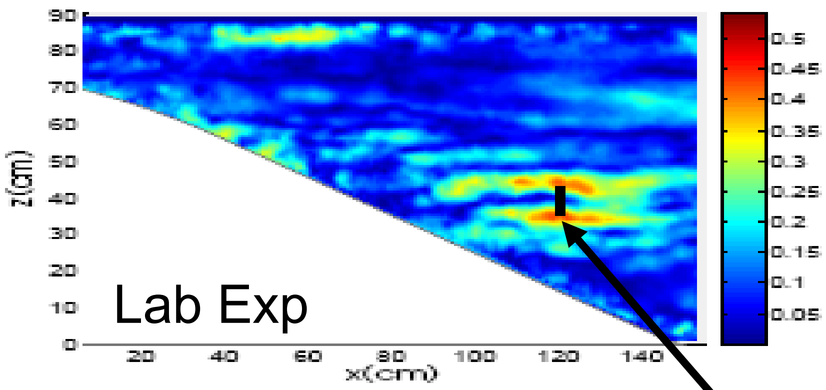
Amplitude of u-component
filtered at M2 frequency (forcing)
(averaged over forcing periods 5 to 7)



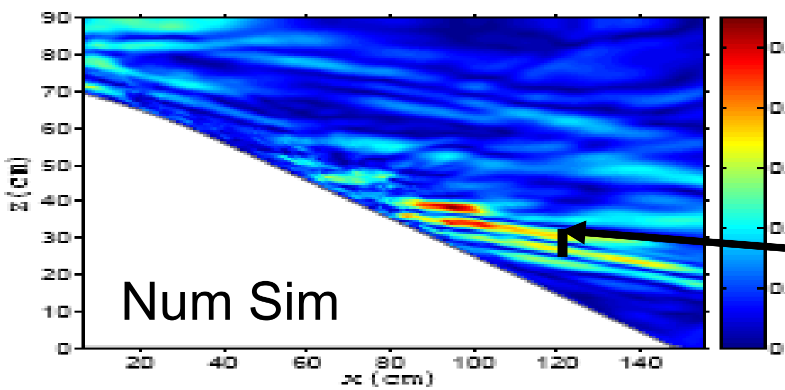
Pairaud et al. 2010

Parametric instability (PSI) of the internal tide beam
No rotation

Amplitude of the horizontal velocity
filtered at half the forcing frequency
over periods 22 to 25

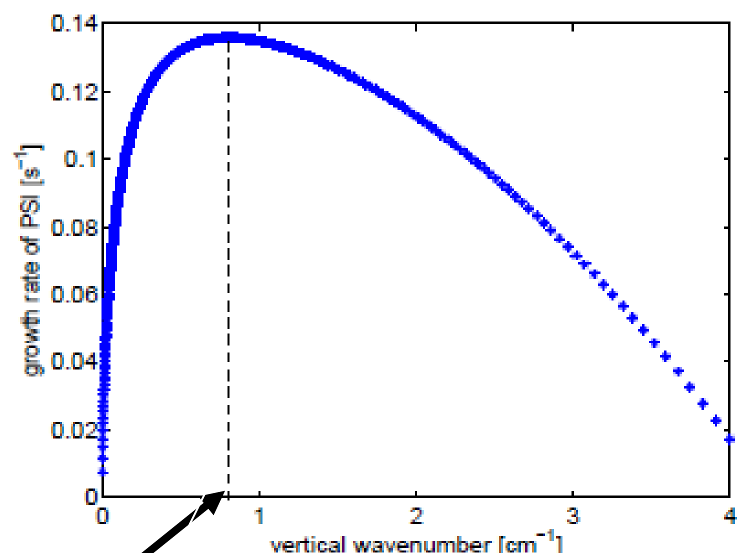


(a)



(b)

Resonant interaction theory
(e.g. Koudella & Staquet JFM 2006)

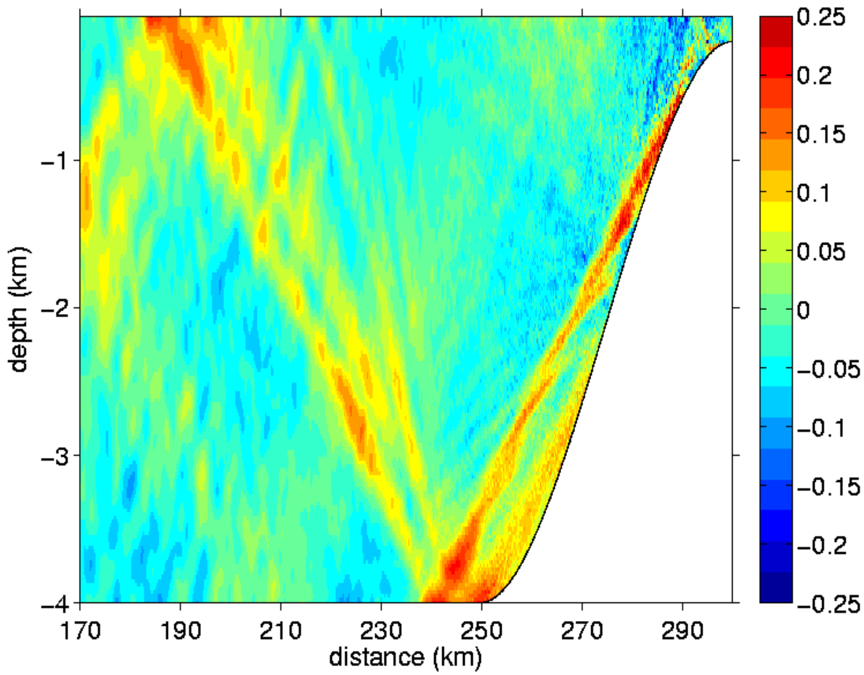


selected scale from PSI

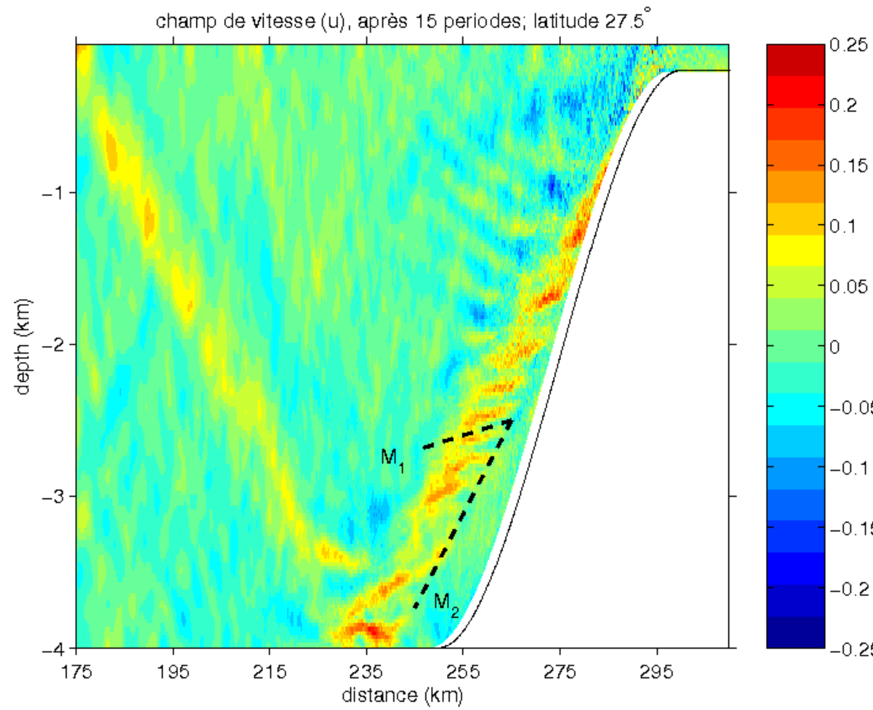
Influence of rotation on an oceanic wave beam

2D numerical simulations : **horizontal velocity component** of the internal tide, at two different latitudes, at 7.5 days after the forcing by the barotropic tide has been applied (no initial velocity field). **The parameter N is constant.**

***f* = 0 (*Φ* = 0º → « equator »)**



***f* = 6.7 10^{-5} rad/s (*Φ* = 27.3º)**

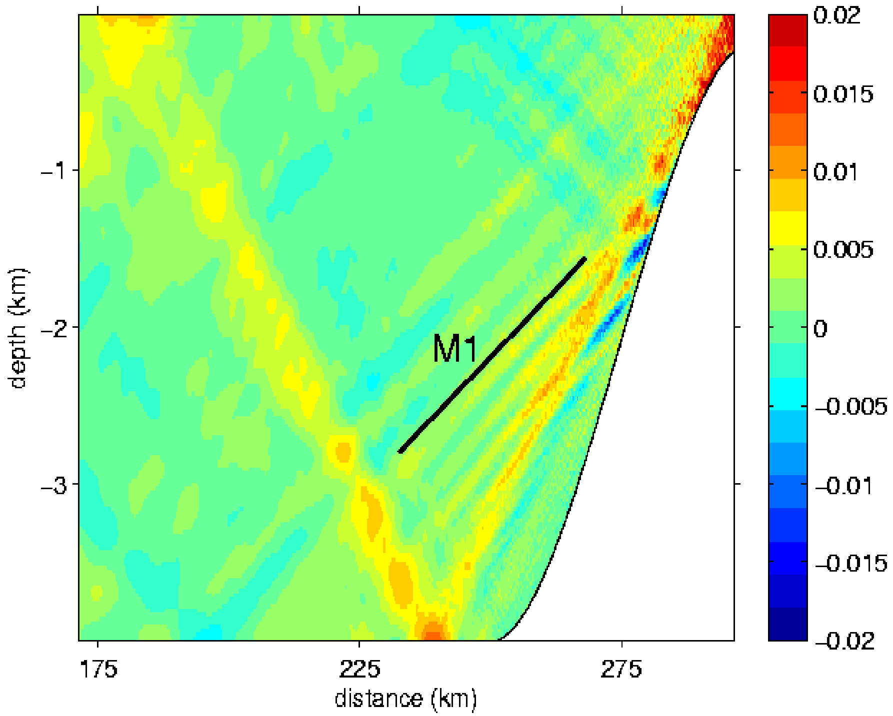


$f = 2\ \Omega \sin\Phi$ is the Coriolis parameter

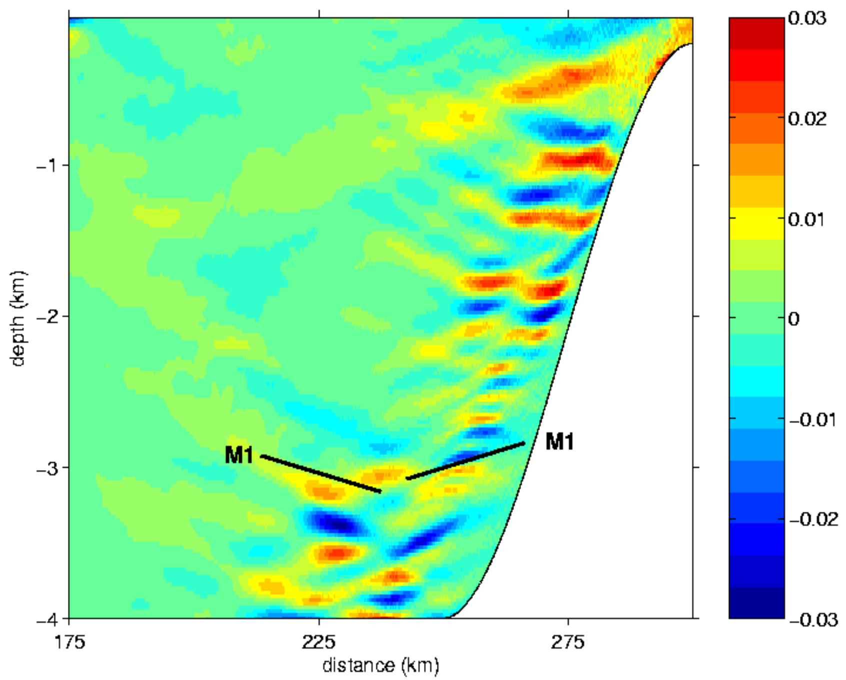
PSI in an oceanic wave beam

Same as previous slide, except that the **horizontal velocity component** has been filtered at **half the forcing frequency** (denoted $\mathbf{M_1}$) to identify the occurrence of a parametric instability.

f = 0 (Φ = 0° → « equator »)

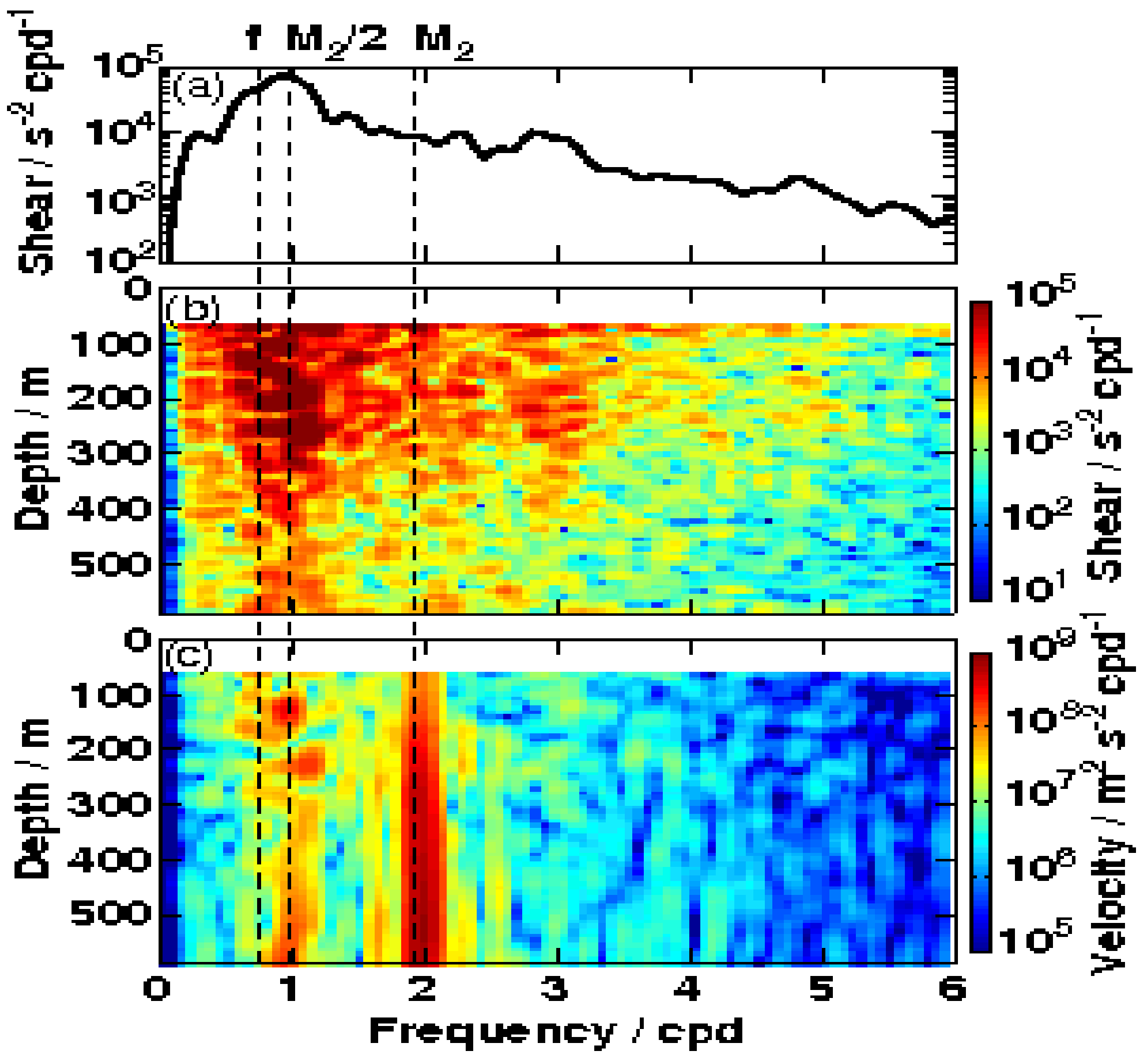


f = 6.7 10⁻⁵ rad/s (Φ = 27.3°)



$f = 2\,\Omega\,\sin\Phi$ is the Coriolis parameter

Parametric instability of the internal tide : oceanic *in situ* evidence



Carter & Gregg 2005 JPO

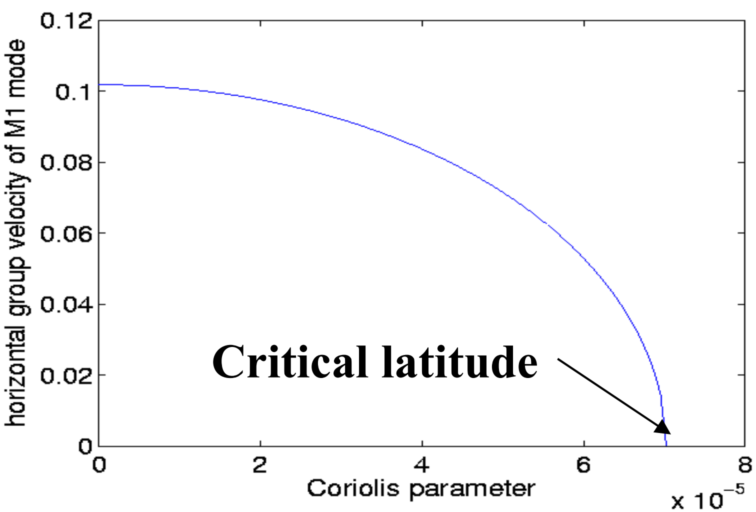
PSI in an internal tide beam (cont'd)

Assuming the wave beam behaves as a monochromatic wave of frequency M_2, it may bear a parametric instability.

Assuming that the perturbation excited by PSI behaves also as a monochromatic wave of frequency M_1 ($=M_2/2$), this frequency satisfies the dispersion relation and is double-bounded: $f \leq M_1 \leq N \Leftrightarrow f \leq M_2/2 \leq N$

Hence, $f = 2\Omega\sin\phi$ has to be smaller than $M_2/2$ for the instability to grow, namely $\phi \leq \phi_c = \mathrm{asin}(0.25\, M_2/\Omega)$, denoted the **critical latitude** and equal to 28.9°.

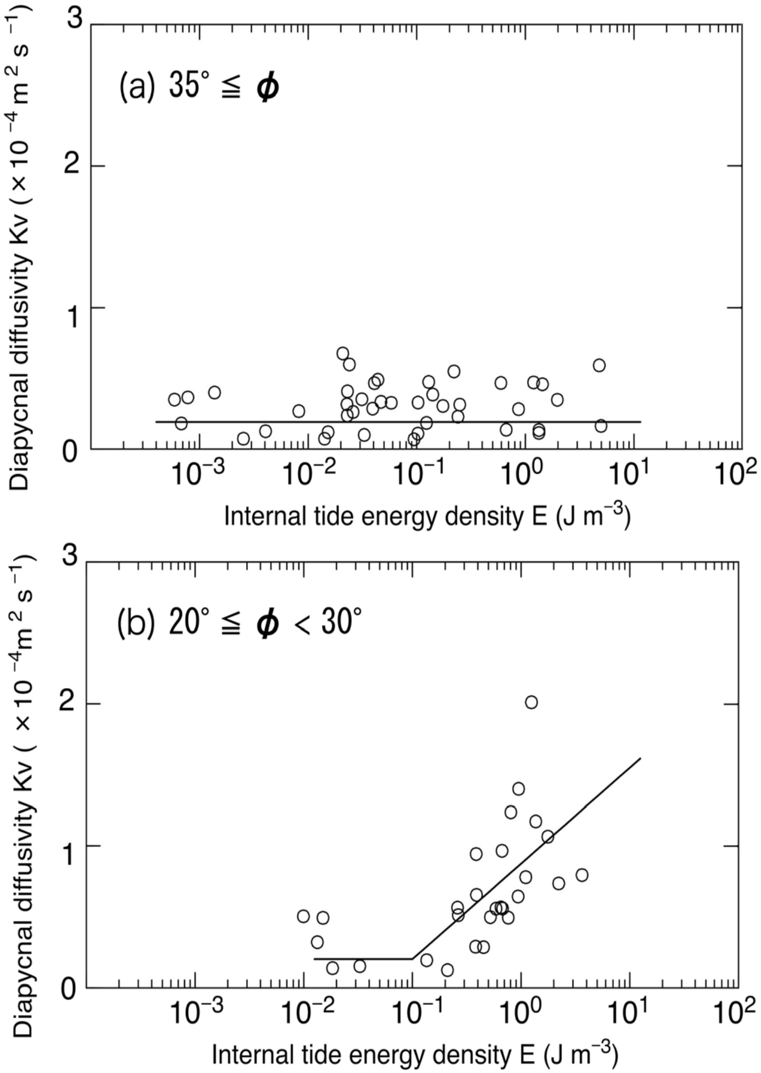
Hence, the M_2 beam can bear a parametric instability for a latitude smaller than 28.9°.



Horizontal group velocity of the M_1 perturbation, versus the Coriolis parameter f.
This group velocity vanishes at the critical latitude → the amplitude of the M1 perturbation is strongest at the critical latitude.

Implication for mixing

Figure : E vs Kd, with
-diapycnal diffusivity estimated from oceanic measurements
-energy density computed numerically at the measurement locations (and averaged over depth 900-1450 m)



Conclusion: Energy is not efficiently transferred toward small scales for $\Phi > 35^o$.
By contrast, a mechanism of energy transfer toward small scales exist for $20^o < \Phi < 30^o$ and this is where PSI has the largest local amplitude.

Hibiya et al 2006

Mixing in the ocean
and PSI in an internal tide beam

Mixing, as measured by a turbulent diffusivity, is largest around the critical latitude, where the perturbation excited by PSI has the largest amplitude.

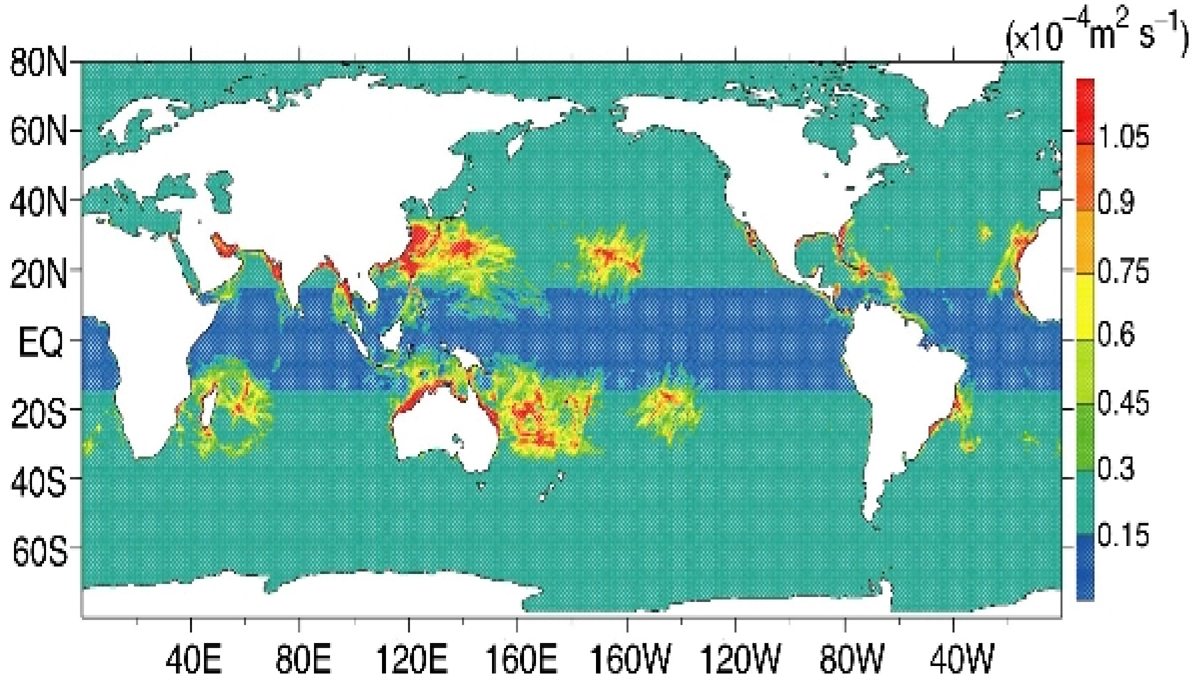
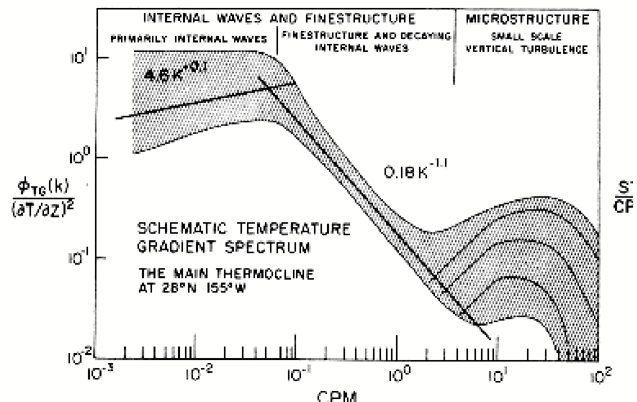


Figure 4. Global distribution of the diapycnal diffusivity calculated by incorporating the numerically-predicted $E(\theta, \phi)$ at each longitude and latitude into the empirical relationship (2).

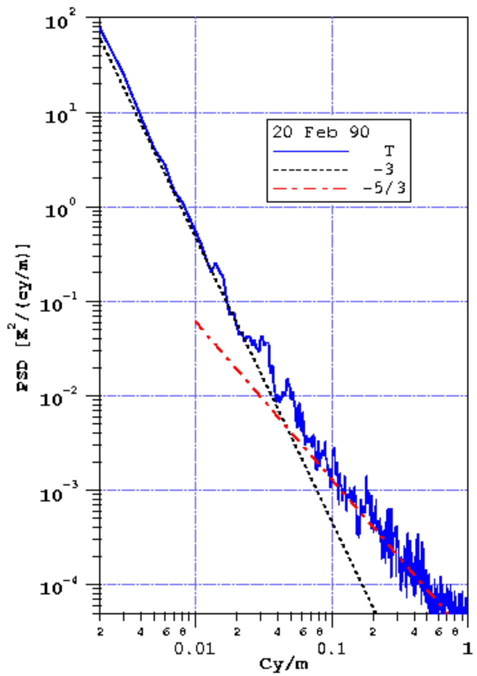
Hibiya et al. GRL 2006

Statistical properties of internal gravity waves

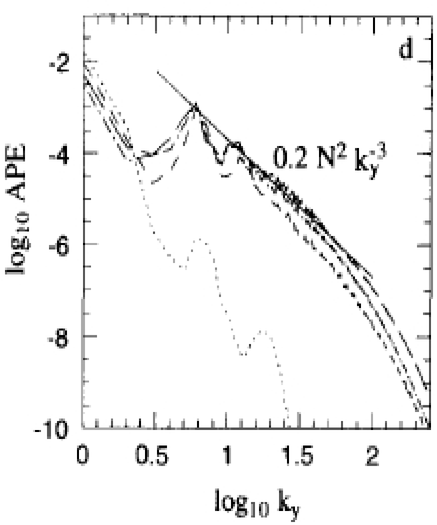
The spectra of nonlinear internal gravity waves : a universal behavior?



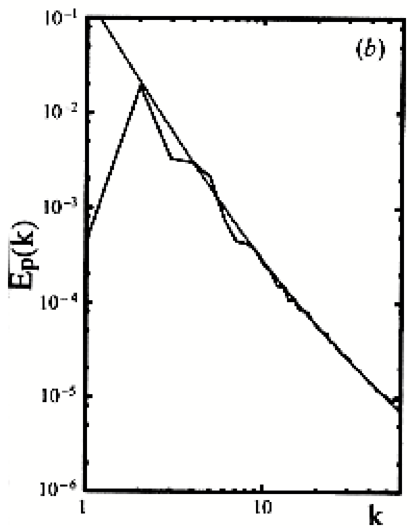
Gregg JGR 1987 Ocean



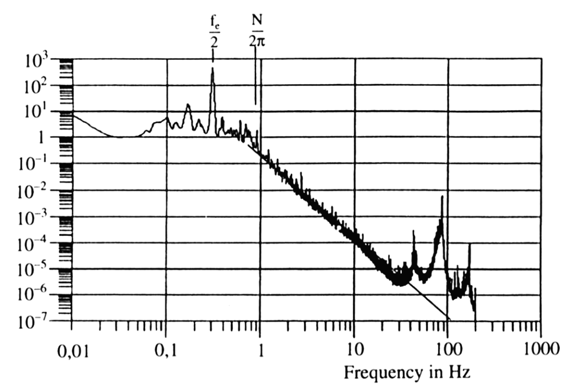
Courtesy F.Dalaudier
Stratosphere



Bouruet-Aubertot et al.
DAO 1996 2D num simul.



Carnevale et al. JFM 2001
3D num. simul.



Benielli & Sommeria
JFM 1998
Laboratory exp.

The spectra of nonlinear internal gravity waves
Cont'd

A simple argument to account for these spectra

One assumes that the amplitude of the waves has saturated namely

$$L/U \approx 1/N \quad (\text{i.e. } Fr = U/NL \approx 1)$$

and that this relation holds at all scales 1/k along the inertial range of the spectra

With $L=1/k$ and $U(k)=[kE(k)]^{1/2}$ one gets

$$k\,[kE(k)]^{1/2} / N \approx 1 \Rightarrow E(k) \approx N^2 k_z^{-3}, \text{ since } k \approx k_z.$$

The factor 0.2 remains to be clarified (mixing efficiency ?)

Conclusions

- Internal gravity waves are propagating motions in a stably stratified fluid, the restoring force being the buoyancy force.
- Internal gravity waves can be unstable to parametric instability (PSI) whatever the stratification level of the fluid, for a wave steepness smaller than ≈ 0.7.
- PSI triggers breaking, which enhances local mixing.
- The internal tide, which are internal gravity waves generated by the tide passing over submarine topography, are thus unstable to PSI but, because rotation influces their dynamics, for a latitude smaller than 28.9°.
- Oceanic mixing is strongest around this latitude, which is attributed to internal tide PSI.
- Breaking internal gravity waves have a universal spectrum of the form $E(k) \approx 0.2\, N^2\, kz^{-3}$, both for the kinetic and potential energy. A simple argument based on wave saturation accounts for these spectra.