LEVEL A

1. Consider a linear system $Ax = b$, where

$$A = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 6 \\ 6 \\ 12 \end{bmatrix}$$

a) Write the matrix formulation of the Gauss-Seidel iterative method for a general matrix A decomposed as $A = L + D + P$. [**7p**]

b) Will Gauss-Seidel method converge for the given system? Why? [**5p**]

c) Choose $x^{(0)} = (0,0,0)^T$ and compute $x^{(1)}$ using Gauss-Seidel method. [**8p**]

d) What is the sufficient and necessary condition for convergence of Gauss-Seidel method? [**5p**]

2. Consider Cauchy problem

$$y''' = \frac{x}{y'+1} + \sqrt{y-3}\,, \qquad y(0) = 4,\ y'(0) = -2,\ y''(0) = 1$$

a) Find a domain G where existence of a unique solution of the problem is guaranteed. [**8p**]

b) Choose step size $h = 0.1$ and compute approximate values of $y(x)$ and $y'(x)$ at $x = 0.2$ using the explicit Euler method. [**17p**]

3. Consider Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = y + 2$$

at the rectangular domain $A = [0,\ 1]$, $B = [1.5,\ 1]$, $C = [1.5,\ 2.5]$, $D = [0,\ 2.5]$ with boundary condition $u(x,y) = 3$ at the boundary of the rectangle ABCD.

a) Which difference formula approximates the second derivative with precision $O(h^2)$? Write down the formula and prove the statement. [**7p**]

b) Derive the scheme for the numerical solution for a general equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ using the finite difference method. [**7p**]

c) Assemble the system of equations for numerical solution of the given problem with step $h = 0.5$. [**11p**]

4. Consider mixed problem for heat equation

$$\frac{\partial u}{\partial t} = 2\,\frac{\partial^2 u}{\partial x^2} + 4x \quad \text{in} \quad \Omega = \{[x,t] : x \in (-1,1),\ t \in (0,T)\}\,,$$

$$u(x,0) = 2x + 2 \text{ for } x \in \langle -1, 1\rangle \text{ and } u(-1,t) = 0,\ u(1,t) = 4 - t \text{ for } t \geq 0.$$

a) Derive the implicit scheme for the equation $\frac{\partial u}{\partial t} = p\frac{\partial^2 u}{\partial x^2} + f(x,t)$. [**6p**]

b) Choose time step $\tau = 0.1$ and space step $h = 0.5$ and assemble equations for numerical solution on the first time level using the implicit scheme. [**10p**]

c) Write the equations from b) in a matrix form. Will Jacobi iteration method converge if used for solving this linear system? What is the reason?[**5p**]

d) Is the implicit scheme for the choice $\tau = 0.1$ and $h = 0.5$ stable? Give reasons for your answer. [**4p**]

LEVEL B

1. Consider a linear system $Ax = b$, where

$$A = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 6 \\ 8 \\ 12 \end{bmatrix}$$

a) Will Jacobi iterative method converge for the given system? Give reasons for your answer. [**6p**]

b) Choose $x^{(0)} = (0,0,0)^T$ and compute $x^{(2)}$ using Jacobi method. [**12p**]

c) Compute the row norm of the difference of the vectors $x^{(1)}$ and $x^{(2)}$. [**7p**]

2. Consider Cauchy problem

$$y' = -x^2 + y\,, \qquad y(0) = 1\,.$$

a) Find an interval I of the maximal solution. [**4p**]

b) Using explicit Euler method with step size $h = 0.5$, compute approximate value of $y(1)$. [**14p**]

c) By means of Taylor expansion, show that the first forward difference approximates the first derivative at a given point with precision $O(h)$. [**7p**]

3. Consider the nonlinear system

$$\begin{aligned} x^2 + y^2 &= 4 \\ y &= 1 - x^2 \end{aligned}$$

a) Find the solution graphically. [**6p**]

b) Choose $x^{(0)} = (2,-2)^T$ and compute $x^{(1)}$ using the Newton's method. [**14p**]

c) Can $x^{(0)} = (0,1)^T$ be choosen as the starting point? Give reason. [**5p**]

4. Consider mixed problem for heat equation

$$\frac{\partial u}{\partial t} = 0.2\,\frac{\partial^2 u}{\partial x^2} + x + t \quad \text{in} \quad \Omega = \{[x,t] : x \in (0,1),\ t \in (0,T)\}\,,$$

$$u(x,0) = x^2, \quad u(0,t) = \sin(t), \quad u(1,t) = \cos(t) \quad .$$

a) Derive the explicit scheme for the equation $\frac{\partial u}{\partial t} = p\frac{\partial^2 u}{\partial x^2} + f(x,t)$. [**8p**]

b) Will the explicit method be stable for a choice of time step $\tau = 0.1$ and space step $h = 0.2$? [**5p**]

c) Choose $\tau = 0.1$ and $h = 0.2$ and compute approximate value of $u(0.2, 0.1)$ using the explicit method. [**12p**]