Numerical solution of differential equations

– example for illustration

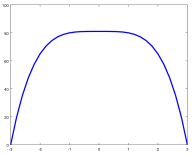
The problem: $-y'' = 12x^2,\ y(-3) = y(3) = 0$

The goal: find the solution $y(x)$ on $\langle -3, 3\rangle$.

This is Poisson equation - can represent stationary heat or diffusion problem (or many others).

Exact solution - integrate twice and use the boundary conditions:

$y(x) = 81 - x^4$



The main steps of numerical solution

A) Discretization

1. Choose a step-size h and construct a mesh:

   $h = 1,\ x_{i+1} = x_i + h:\quad -3 = x_0 < x_1 < \ldots x_6 = 3$

2. Put together algebraic equations for unknowns $y_i \approx y(x_i)$
   – approximate values of the solution at x_i

   2.1 Express the equation at every x_i inside $(-3, 3)$:

   $$-y''(x_i) = 12x_i^2\,,\quad i = 1, \ldots 5$$

   2.2 Substitute differences instead of derivatives:

   $$y''(x_i) \approx \frac{y(x_{i-1}) - 2y(x_i) + y(x_{i+1})}{h^2} \approx \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

   to get a system of 5 (lin.) equations for 5 unknowns $y_1, \ldots y_5$:

   $$-y_{i-1} + 2y_i - y_{i+1} = 12x_i^2\,,\quad i = 1, \ldots 5,\ y_0 = y_6 = 0$$

   after substitution of values, in matrix form:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 48 \\ 12 \\ 0 \\ 12 \\ 48 \end{bmatrix}$$

B) Solving a system of algebraic equations

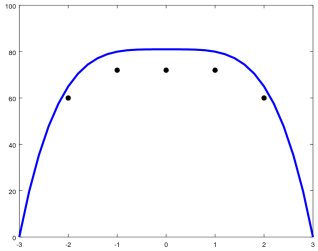
in our example we have linear system $AY = B$, where

A ... an approximation of the given differential operator on some mesh,
Y ... an approximation of values of the exact solution at mesh-points,
B ... the right side of the equation and (Dirichlet) boundary conditions.

The numerical solution Y compared with exact solution $y(x_i)$:

$$Y = \begin{bmatrix} 60 \\ 72 \\ 72 \\ 72 \\ 60 \end{bmatrix} , \quad Y_{ex} = \begin{bmatrix} y(x_1) \\ y(x_2) \\ y(x_3) \\ y(x_4) \\ y(x_5) \end{bmatrix} = \begin{bmatrix} 65 \\ 80 \\ 81 \\ 80 \\ 65 \end{bmatrix} , \quad Y_{ex} - Y = \begin{bmatrix} 5 \\ 8 \\ 9 \\ 8 \\ 5 \end{bmatrix}$$

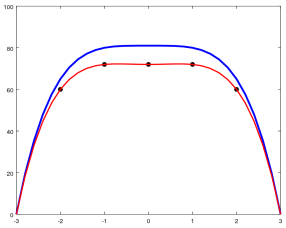
– about 10% error – not so bad on such a coarse mesh



Questions:

- Is the matrix A always regular?
- Does $Y \to Y_{ex}$ as $h \to 0$?
- How quick is the convergence?
- ...

C) Interpolation, approximation of discrete results by a continuous function (often not so important):



here the least square method for a polynomial of the 4-th degree was used

Our roadmap

We are going to learn:

1. Some methods for **solving a system of algebraic equations**
2. **Interpolation, approximation** - minor topic (1 week)
3. **Discretization of ODR, PDR** - the main theme