Finite differences for second order linear PDE in 2 variables

PDE are classified into three types:

- **elliptic** (example: Poisson equation)
- **parabolic** (example: heat equation)
- **hyperbolic** (example: wave equation)

Discretization of PDE (inside the given domain) consists of the three following steps:

1. Choosing the step-size in both directions and constructing the grid.
2. Expressing the equation at every grid node (inside the domain).
3. Substitution of derivatives with the finite differences.

Caution: All terms of the equation have to be expressed or approximated at the same grid node.

Wave equation

Mixed problem for wave equation

We are seeking a function $u \equiv u(x,t)$ which satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + f(x,t) \quad \text{in the domain } \Omega = (a,b) \times (0,T) \ , \tag{1}$$

has prescribed initial conditions at time $t = 0$

$$u(x,0) = \phi(x) \ \text{ for } x \in \langle a,b \rangle \ , \qquad \frac{\partial u}{\partial t}(x,0) = \psi(x) \ \text{ for } x \in \langle a,b \rangle$$

and has prescribed boundary values for $t > 0$: $\quad u(a,t) = \alpha(t)$, $\quad u(b,t) = \beta(t)$.
Wave speed c is supposed to be a positive constant.

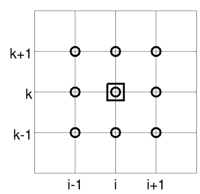
Initial and boundary conditions have to satisfy *conditions of compatibility*:

$$\phi(a) = \alpha(0) \ , \quad \phi(b) = \beta(0) \ , \quad \psi(a) = \alpha'(0) \ , \quad \psi(b) = \beta'(0) \ .$$

Discretization:

1. Variables x and t in the equation represent different entities (x usually represent spatial direction and t represents time). So it is natural to choose different step-sizes and construct a rectangular grid of nodes over Ω with equal mesh spacing h in x direction and τ for t.

Scheme of the grid around a grid node P_i^k:



Notation:

$P_i^k \equiv [x_i, t_k]$ ... grid nodes, where
x_i ... x-coordinates of the nodes: $h = x_{i+1} - x_i$
t_k ... t-coordinates of the nodes: $\tau = t_{k+1} - t_k$

$u(x,t)$ ... function of two variables defined in Ω, $u(P_i^k) \equiv u(x_i, t_k)$

$U_i^k \approx u(P_i^k)$ ... approximate value of $u(x,t)$ at a grid node P_i^k

Explicit method

2. Express the equation (1) at every node $P_i^k = [x_i, t_k],\ k = 0, 1, \ldots$:

$$\frac{\partial^2 u}{\partial t^2}(P_i^k) \ = \ c^2 \, \frac{\partial^2 u}{\partial x^2}(P_i^k) + f(P_i^k) \tag{2}$$

3. Use the second central differences at the node P_i^k for approximation of the second derivatives with respect to x and t, respectively (see Figure 1):

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(P_i^k) &= \frac{u(P_{i-1}^k) - 2u(P_i^k) + u(P_{i+1}^k)}{h^2} + \mathcal{O}(h^2) \approx \frac{U_{i-1}^k - 2U_i^k + U_{i+1}^k}{h^2} \\
\frac{\partial^2 u}{\partial t^2}(P_i^k) &= \frac{u(P_i^{k-1}) - 2u(P_i^k) + u(P_i^{k+1})}{\tau^2} + \mathcal{O}(\tau^2) \approx \frac{U_i^{k-1} - 2U_i^k + U_i^{k+1}}{\tau^2}
\end{aligned}$$

Substitution of these differences into (2):

$$\frac{U_i^{k-1} - 2U_i^k + U_i^{k+1}}{\tau^2} \ = \ c^2 \, \frac{U_{i-1}^k - 2U_i^k + U_{i+1}^k}{h^2} + f(P_i^k) \ .$$

After rearrangig this leads to equation for 5 unknowns:

$$U_i^{k+1} = \sigma^2 \, U_{i-1}^k + 2(1 - \sigma^2) \, U_i^k + \sigma^2 \, U_{i+1}^k - U_i^{k-1} + \tau^2 \, f(P_i^k), \tag{3}$$

where $\sigma = \dfrac{c\,\tau}{h}$.

Condition of stability for explicit method: $\sigma \leq 1$.

Numerical solution is evaluated one time level after another: from known values at $(k-1)$-st and k-th time level, values at $(k+1)$-st time level are computed, one node after another, using the formula (3). Values at left and right boundaries are given by the boundary conditions $\alpha(t)$ and $\beta(t)$, respectively. Values at the initial time level are given by the initial condition $\phi(x)$ and values at the first time level are extrapolated from initial time level as

$$U_i^1 = \phi(x_i) + \tau \psi(x_i) \ .$$

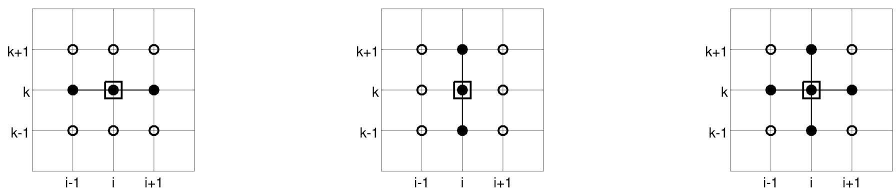


Figure 1: Grid nodes used for finite differences centered at the node P_i^k. Left: 2-nd central difference with respect to x. Center: 2-nd central difference with respect to t. Right: Five-point stencil for explicit method for wave equation.

Implicit method

2. Express the equation (1) at every node $P_i^k = [x_i, t_k],\ k = 0, 1, \ldots$:

$$\frac{\partial^2 u}{\partial t^2}(P_i^k) \ = \ c^2\,\frac{\partial^2 u}{\partial x^2}(P_i^k) + f(P_i^k) \tag{4}$$

3. Use the second central difference for approximation of the partial derivative with respect to t at the node P_i^k as

$$\frac{\partial^2 u}{\partial t^2}(P_i^k) = \frac{u(P_i^{k-1}) - 2u(P_i^k) + u(P_i^{k+1})}{\tau^2} + \mathcal{O}(\tau^2) \approx \frac{U_i^{k-1} - 2U_i^k + U_i^{k+1}}{\tau^2}$$

as before. For approximation of the second partial derivative with respect to x, use the formula

$$\frac{\partial^2 u}{\partial x^2}(P_i^k) = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}(P_i^{k-1}) + \frac{\partial^2 u}{\partial x^2}(P_i^{k+1})\right) + \mathcal{O}(\tau^2)$$

and then use the second central difference for approximation of the partial derivative with respect to x for both terms on the right hand side.

After substitutig these differences into (4) and rearrangig this leads to equation for 5 unknowns:

$$-\sigma^2 U_{i-1}^{k+1} + 2(1+\sigma^2)\,U_i^{k+1} - \sigma^2 U_{i+1}^{k+1} = \sigma^2 U_{i-1}^{k-1} - 2(1+\sigma^2)\,U_i^{k-1} + \sigma^2 U_{i+1}^{k-1} + 4U_i^k + 2\tau^2\,f(P_i^k) \tag{5}$$

where $\sigma = \dfrac{c\,\tau}{h}$.

The discretization is performed at every inner node of the k-th time level, so a system of linear equations is obtained, from which values at the $(k+1)$-st time level can be computed. Values at the two initial time levels are given by the initial conditions, values at left and right boundaries are given by the boundary conditions.

Implicit scheme is **unconditionally stable.**

Problem

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\,\frac{\partial^2 u}{\partial x^2} + 2x \qquad \text{in domain } \Omega = (-1,1)\times(0,T)$$

with initial conditions

$u(x,0) = 2x^2\ ,\ \ \frac{\partial u}{\partial t}(x,0) = (1-x)\,\sin(\frac{\pi\,x}{2}) \quad$ for $x \in \langle -1, 1\rangle$

and boundary conditions $\quad u(-1,t) = k\,\mathrm{e}^{-t}\ ,\ \ u(1,t) = 2$ for $t > 0$.

a) Find the value of the parameter k so that the initial and boundary conditions are compatible.

b) Compute the approximate value of the solution at $x = 0.8$ and $t = 0.24$ using the explicit finite difference method. Choose $h = 0.2$ and choose τ as big as possible, provided it still leads to the stable explicit method.

Solution

a) Compatibility at the corner [-1,0]:

initial condition:

$u(-1,0) = 2x^2|_{x=-1} = 2\cdot(-1)^2 = 2$

$\frac{\partial u}{\partial t}(-1,0) = (1-x)\,\sin(\frac{\pi\,x}{2})|_{x=-1} = (1-(-1))\,\sin(\frac{\pi\cdot(-1)}{2}) = 2\cdot(-1) = -2$

boundary condition:

$u(-1,0) = k\,\mathrm{e}^{-t}|_{t=0} = k\,\mathrm{e}^{0} = k$

$\frac{\partial u}{\partial t}|_{t=0} = -k\,\mathrm{e}^{-t}|_{t=0} = -k$

the conditions are compatible $\quad\Leftrightarrow\quad k = 2$.

b) The time step-size τ has to fulfill $\sigma = \frac{c\,\tau}{h} \le 1$, i.e. $\tau \le 1\cdot\frac{h}{c} = \frac{0.2}{\sqrt{4}} = 0.1$. Choose the maximal $\tau \le 0.1$ such that the point [0.8, 0.24] is a mesh node:

$\tau = 0.08$, so $\sigma = \frac{c\,\tau}{h} = \frac{2\cdot 0.08}{0.2} = 0.8$.

Let us prepare a table, which then will be subsequently filled by rows from the initial time level, as the time levels are computed one after another. There are only the values necessary for computation designated in the table. The main difference from the solution of the heat equation is this: now we have to use the initial conditions for determining not only the initial time level (blue), but also the first one (green). The table layout:

			. . .					
t_3	0.24		. . .				U_9^3	
t_2	0.16		. . .			U_8^2	U_9^2	U_{10}^2
t_1	0.08		. . .		U_7^1	U_8^1	U_9^1	U_{10}^1
t_0	0.0		. . .		U_7^0	U_8^0	U_9^0	U_{10}^0
		-1.0	. . .	0.2	0.4	0.6	0.8	1.0
		x_0	. . .	x_6	x_7	x_8	x_9	x_{10}

Let us start by computing the initial and the first time levels from the initial conditions and the last column from the boundary condition:

$U_{10}^1 = u(1,t_1) = 2,\ \ U_{10}^2 = u(1,t_2) = 2$

$U_7^0 = u(x_7, 0) = 2 \cdot x_7^2 = 2 \cdot 0.4^2 = 0.32$
$U_8^0 = u(x_8, 0) = 2 \cdot 0.6^2 = 0.72, \;\; U_9^0 = u(x_9, 0) = 2 \cdot 0.8^2 = 1.28$
$U_{10}^0 = u(x_{10}, 0) = 2 \cdot 1^2 = 2 \;\; (= u(1,0) \ldots \text{compatibility condition})$

$$U_7^1 = U_7^0 + \tau \frac{\partial u}{\partial t}(x_7, 0) = U_7^0 + \tau(1 - x_7)\,\sin(\frac{\pi\, x_7}{2}) =$$
$$= 0.32 + 0.08 \cdot (1 - 0.4)\,\sin(\frac{0.4\,\pi}{2}) = 0.3482$$
$$U_8^1 = U_8^0 + \tau \frac{\partial u}{\partial t}(x_8, 0) = 0.72 + 0.08 \cdot (1 - 0.6)\,\sin(\frac{0.6\,\pi}{2}) = 0.7459$$
$$U_9^1 = U_9^0 + \tau \frac{\partial u}{\partial t}(x_9, 0) = 1.28 + 0.08 \cdot (1 - 0.8)\,\sin(\frac{0.8\,\pi}{2}) = 1.2952$$

Now let us subsequently compute values at particular time level, using the already computed values from the two previous time levels:

The second level ($t_2 = 0.16$):

$$U_8^2 = 2 \cdot (1 - \sigma^2)\, U_8^1 + \sigma^2 (U_7^1 + U_9^1) - U_8^0 + \tau^2 f(x_8, t_1) =$$
$$= 2 \cdot (1 - 0.64) \cdot 0.7459 + 0.64\,(0.3482 + 1.2952) - 0.72 + 0.0064 \cdot (2 \cdot 0.6) =$$
$$= 0.8765$$
$$U_9^2 = 2 \cdot (1 - \sigma^2)\, U_9^1 + \sigma^2 (U_8^1 + U_{10}^1) - U_9^0 + \tau^2 f(x_9, t_1) =$$
$$= 2 \cdot 0.36 \cdot 1.2952 + 0.64\,(0.7459 + 2) - 1.28 + 0.0064 \cdot (2 \cdot 0.8) = 1.4202$$

The third level ($t_3 = 0.24$):

$$U_9^3 = 2 \cdot (1 - \sigma^2)\, U_9^2 + \sigma^2 (U_8^2 + U_{10}^2) - U_9^1 + \tau^2 f(x_9, t_2) =$$
$$= 2 \cdot 0.36 \cdot 1.4202 + 0.64\,(0.8765 + 2) - 1.2952 + 0.0064 \cdot (2 \cdot 0.8) = 1.5785$$

			...					
t_3	0.24		...				1.5785	
t_2	0.16		...			0.8765	1.4202	2.0000
t_1	0.08		...		0.3482	0.7459	1.2952	2.0000
t_0	0.0		...		0.3200	0.7200	1.2800	2.0000
		-1.0	...	0.2	0.4	0.6	0.8	1.0
		x_0	...	x_6	x_7	x_8	x_9	x_{10}

The approximate value of $u(0.8, 0.24)$ is $U_9^3 = 1.5785$.