Partial differential equations III

The theory (very short excerpts from the lectures)

Mixed problem for wave equation

We are seeking a function $u \equiv u(x,t)$ which satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + f(x,t) \quad \text{in the domain } \Omega = (a,b) \times (0,T) \ ,$$

has prescribed initial conditions at time $t = 0$

$$u(x,0) = \phi(x) \quad \text{for } x \in \langle a,b \rangle$$
$$\frac{\partial u}{\partial t}(x,0) = \psi(x) \quad \text{for } x \in \langle a,b \rangle$$

and has prescribed boundary values for $t > 0$

$$u(a,t) = \alpha(t) \ , \quad u(b,t) = \beta(t)$$

Wave speed c is supposed to be a positive constant.

Initial and boundary conditions have to obey *conditions of compatibility*:

$$\phi(a) = \alpha(0) \ , \quad \phi(b) = \beta(0) \ , \quad \psi(a) = \alpha'(0) \ , \quad \psi(b) = \beta'(0)$$

Finite difference method

- Discretize the domain Ω as in the previous problem of heat equation. Put $\sigma = \dfrac{c\,\tau}{h}$.
- Values at left and right boundaries are given by the boundary conditions, values at the initial time level are given by the initial condition $\phi(x)$ and values at the first time level are extrapolated from initial time level as
  $$U_i^{(1)} = \phi(x_i) + \tau\psi(x_i) \ .$$
- **Explicit scheme**: At every interior node $P_i^{(k+1)}$, compute approximate value $U_i^{(k+1)}$ from values $U_{i-1}^{(k)}$, $U_i^{(k)}$, $U_{i+1}^{(k)}$, $U_i^{(k-1)}$ at the previous two time levels as
  $$U_i^{(k+1)} = \sigma^2 U_{i-1}^{(k)} + 2(1-\sigma^2)\, U_i^{(k)} + \sigma^2 U_{i+1}^{(k)} - U_i^{(k-1)} + \tau^2 f(x_i,t_k) \ .$$
  Condition of stability: $\quad \sigma \le 1$.
- **Implicit scheme**: Compute all values $U_i^{(k+1)}$ at interior nodes in $(k+1)$-th time level from values $U_j^{(k)}$, $U_j^{(k-1)}$ at the previous two time levels from a system of linear equations
  $$-\sigma^2 U_{i-1}^{(k+1)} + 2(1+\sigma^2)\, U_i^{(k+1)} - \sigma^2 U_{i+1}^{(k+1)} =$$
  $$= \sigma^2 U_{i-1}^{(k-1)} - 2(1+\sigma^2)\, U_i^{(k-1)} + \sigma^2 U_{i+1}^{(k-1)} + 4U_i^{(k)} + 2\tau^2 f(x_i,t_k) \ .$$
  Implicit scheme is unconditionally stable.

Problem 1

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\,\frac{\partial^2 u}{\partial x^2} + 2x \qquad\qquad \text{in domain } \Omega = (-1,1)\times(0,T)$$

with initial conditions

$u(x,0) = 2x^2$, $\dfrac{\partial u}{\partial t}(x,0) = (1-x)\,\sin(\dfrac{\pi\,x}{2})$ for $x \in \langle -1, 1\rangle$

and boundary conditions $u(-1,t) = k\,\mathrm{e}^{-t}$, $u(1,t) = 2$ for $t > 0$.

a) Find the value of the parameter k so that the initial and boundary conditions are compatible.

b) Compute the approximate value of the solution at $x = 0.8$ and $t = 0.24$ using the explicit finite difference method. Choose $h = 0.2$ and choose τ as big as possible, provided it still leads to the stable explicit method.

The solution

a) Compatibility at the corner [-1,0]:

initial condition:

$u(-1,0) = 2x^2|_{x=-1} = 2\cdot(-1)^2 = 2$

$\dfrac{\partial u}{\partial t}(-1,0) = (1-x)\,\sin(\dfrac{\pi\,x}{2})|_{x=-1} = (1-(-1))\,\sin(\dfrac{\pi\cdot(-1)}{2}) = 2\cdot(-1) = -2$

boundary condition:

$u(-1,0) = k\,\mathrm{e}^{-t}|_{t=0} = k\,\mathrm{e}^{0} = k$

$\dfrac{\partial u}{\partial t}|_{t=0} = -k\,\mathrm{e}^{-t}|_{t=0} = -k$

the conditions are compatible $\Leftrightarrow$ $k = 2$.

b) The time step-size τ has to fulfill $\sigma = \dfrac{c\,\tau}{h} \le 1$, i.e. $\tau \le 1\cdot\dfrac{h}{c} = \dfrac{0.2}{\sqrt{4}} = 0.1$.
Choose the maximal $\tau \le 0.1$ such that the point [0.8, 0.24] is a mesh node:

$\tau = 0.08$, so $\sigma = \dfrac{c\,\tau}{h} = \dfrac{2\cdot 0.08}{0.2} = 0.8$.

Let us prepare a table, which then will be subsequently filled by rows from the initial time level, as the time levels are computed one after another. There are only the values necessary for computation designated in the table. The main difference from the solution of the heat equation is, that now we have to use the initial conditions for determining not only the initial time level (blue), but also the first one (green). The table layout:

			$\dots$					
t_3	0.24		$\dots$				U_9^3	
t_2	0.16		$\dots$			U_8^2	U_9^2	U_{10}^2
t_1	0.08		$\dots$		U_7^1	U_8^1	U_9^1	U_{10}^1
t_0	0.0		$\dots$		U_7^0	U_8^0	U_9^0	U_{10}^0
		-1.0	$\dots$	0.2	0.4	0.6	0.8	1.0
		x_0	$\dots$	x_6	x_7	x_8	x_9	x_{10}

Let us start by computing the initial and the first time levels from the initial conditions and the last column from the boundary condition:

$U_{10}^1 = u(1, t_1) = 2, \;\; U_{10}^2 = u(1, t_2) = 2$

$U_7^0 = u(x_7, 0) = 2 \cdot x_7^2 = 2 \cdot 0.4^2 = 0.32$

$U_8^0 = u(x_8, 0) = 2 \cdot 0.6^2 = 0.72, \;\; U_9^0 = u(x_9, 0) = 2 \cdot 0.8^2 = 1.28$

$U_{10}^0 = u(x_{10}, 0) = 2 \cdot 1^2 = 2 \;\; (= u(1,0) \dots \text{compatibility condition})$

$$U_7^1 = U_7^0 + \tau \frac{\partial u}{\partial t}(x_7, 0) = U_7^0 + \tau(1 - x_7)\,\sin(\frac{\pi\, x_7}{2}) = $$
$$= 0.32 + 0.08 \cdot (1 - 0.4)\,\sin(\frac{0.4\,\pi}{2}) = 0.3482$$

$$U_8^1 = U_8^0 + \tau \frac{\partial u}{\partial t}(x_8, 0) = 0.72 + 0.08 \cdot (1 - 0.6)\,\sin(\frac{0.6\,\pi}{2}) = 0.7459$$

$$U_9^1 = U_9^0 + \tau \frac{\partial u}{\partial t}(x_9, 0) = 1.28 + 0.08 \cdot (1 - 0.8)\,\sin(\frac{0.8\,\pi}{2}) = 1.2952$$

Now let us subsequently compute values at particular time level, using the already computed values from the two previous time levels:

The second level ($t_2 = 0.16$):

$$U_8^2 = 2 \cdot (1 - \sigma^2)\,U_8^1 + \sigma^2(U_7^1 + U_9^1) - U_8^0 + \tau^2\,f(x_8, t_1) =$$
$$= 2 \cdot (1 - 0.64) \cdot 0.7459 + 0.64\,(0.3482 + 1.2952) - 0.72 + 0.0064 \cdot (2 \cdot 0.6) =$$
$$= 0.8765$$

$$U_9^2 = 2 \cdot (1 - \sigma^2)\,U_9^1 + \sigma^2(U_8^1 + U_{10}^1) - U_9^0 + \tau^2\,f(x_9, t_1) =$$
$$= 2 \cdot 0.36 \cdot 1.2952 + 0.64\,(0.7459 + 2) - 1.28 + 0.0064 \cdot (2 \cdot 0.8) = 1.4202$$

The third level ($t_3 = 0.24$):

$$U_9^3 = 2 \cdot (1 - \sigma^2)\,U_9^2 + \sigma^2(U_8^2 + U_{10}^2) - U_9^1 + \tau^2\,f(x_9, t_2) =$$
$$= 2 \cdot 0.36 \cdot 1.4202 + 0.64\,(0.8765 + 2) - 1.2952 + 0.0064 \cdot (2 \cdot 0.8) = 1.5785$$

			$\dots$					
t_3	0.24		$\dots$				1.5785	
t_2	0.16		$\dots$			0.8765	1.4202	2.0000
t_1	0.08		$\dots$		0.3482	0.7459	1.2952	2.0000
t_0	0.0		$\dots$		0.3200	0.7200	1.2800	2.0000
		-1.0	$\dots$	0.2	0.4	0.6	0.8	1.0
		x_0	$\dots$	x_6	x_7	x_8	x_9	x_{10}

The approximate value of $u(0.8, 0.24)$ is $U_9^3 = 1.5785$.