Partial differential equations II

The theory (very short excerpts from the lectures)

Mixed problem for heat equation

We are seeking a function $u \equiv u(x,t)$ which satisfies

$$\frac{\partial u}{\partial t} = p\,\frac{\partial^2 u}{\partial x^2} + f(x,t) \quad \text{in the domain } \Omega = (a,b) \times (0,T) \ ,$$

has prescribed initial condition at time $t = 0$

$u(x,0) = \phi(x)$ for $x \in \langle a, b\rangle$

and has prescribed boundary values for $t > 0$

$u(a,t) = \alpha(t) \ , \quad u(b,t) = \beta(t) \ .$

Coefficient of thermal diffusivity p is supposed to be constant.

Initial and boundary conditions have to obey *conditions of compatibility*:

$\phi(a) = \alpha(0) \ , \quad \phi(b) = \beta(0)$

Finite difference method

- Discretize the domain Ω by a grid of uniformly spaced nodes with stepsizes h and τ in the x and t directions, respectively. Denote the nodes as $P_i^{(k)} = [x_i, t_k]$, where $a = x_0 < x_1 < \cdots < x_n = b$, $x_{i+1} = x_i + h$, $0 = t_0 < t_1 < \ldots, \quad t_{k+1} = t_k + \tau$. Denote as $U_i^{(k)}$ the approximate value of the solution at the node $P_i^{(k)}$, i.e. $U_i^{(k)} \approx u(x_i, t_k)$. Put $\sigma = \dfrac{p\,\tau}{h^2}$.
- Values at the initial time level are given by the initial condition, values at left and right boundaries are given by the boundary conditions.
- **Explicit scheme**: At every interior node $P_i^{(k+1)}$, compute approximate value $U_i^{(k+1)}$ from values $U_{i-1}^{(k)}$, $U_i^{(k)}$, $U_{i+1}^{(k)}$ at the previous time level as

  $$U_i^{(k+1)} = \sigma U_{i-1}^{(k)} + (1-2\sigma)\,U_i^{(k)} + \sigma U_{i+1}^{(k)} + \tau\,f(x_i,t_k) \ .$$

  Condition of stability: $\sigma \le 0.5$.
- **Implicit scheme**: Compute all values $U_i^{(k+1)}$ at interior nodes in the $(k+1)$-th time level from values $U_j^{(k)}$ at the previous time level from a system of linear equations

  $$-\sigma U_{i-1}^{(k+1)} + (1+2\sigma)\,U_i^{(k+1)} - \sigma U_{i+1}^{(k+1)} = U_i^{(k)} + \tau\,f(x_i,t_{k+1}) \ .$$

  Implicit scheme is unconditionally stable.

Problem 1

Find the solution of the heat equation

$$\frac{\partial u}{\partial t} = 0.3\,\frac{\partial^2 u}{\partial x^2} + x \quad \text{in domain } \Omega = (0,1) \times (0,0.4)$$

with initial condition $\quad u(x,0) = x^2 \;$ for $x \in \langle 0,1\rangle$

and boundary conditions $\quad u(0,t) = 0\;,\; u(1,t) = 1$ for $t > 0$.

a) Choose the spatial step-size $h = 0.25$ and use the time step-size τ as big as possible, provided it still leads to the stable explicit method. Then use the explicit method.

b) Choose the step-size h and the time step-size twice as big as before and use the implicit method.

The solution

First of all, let us check the compatibility of the initial and boundary conditions: both conditions are equal to zero for $x = 0$, $t = 0$ and both conditions are equal to one for $x = 1$, $t = 0$. We can see that the initial and boundary conditions are compatible.

a) We are seeking the maximal time step-size τ such that the explicit method is stable, i.e. $\sigma \le 0.5$:

$$\sigma = \frac{p\,\tau}{h^2} \le 0.5 \quad \Leftrightarrow \quad \tau \le 0.5\,\frac{h^2}{p} = 0.5\,\frac{0.25^2}{0.3} = 0.10417$$

In order to solve the problem within the time interval $\langle 0, 0.4\rangle$, choose τ such that the end time value 0.4 is its multiple: set $\tau = 0.1$, then

$$\sigma = \frac{0.3 \cdot 0.1}{0.25^2} = 0.48\ .$$

Let us prepare a table, which then will be subsequently filled by rows, starting from the initial time level (in the bottom). The layout of the table:

t_4	0.4	U_0^4	U_1^4	U_2^4	U_3^4	U_4^4
t_3	0.3	U_0^3	U_1^3	U_2^3	U_3^3	U_4^3
t_2	0.2	U_0^2	U_1^2	U_2^2	U_3^2	U_4^2
t_1	0.1	U_0^1	U_1^1	U_2^1	U_3^1	U_4^1
t_0	0.0	U_0^0	U_1^0	U_2^0	U_3^0	U_4^0
		0.00	0.25	0.50	0.75	1.00
		x_0	x_1	x_2	x_3	x_4

Values of t are in the first column of the table, x-coordinate nodes are in the bottom row. We start by filling the values given by the initial condition into the row for $t = 0$ (blue) and the values given by the boundary conditions into the second and the last columns (red). Values at the corners (violet) should be the same whether they are computed from the initial condition or from the boundary condition. Values of the solution we are searching for are inside the table (black). In the beginning, after filling in the initial condition $u(x, 0) = x^2$ and the boundary conditions $u(0, t) = 0$ and $u(1, t) = 1$, we have

t_4	0.4	0.0000				1.0000
t_3	0.3	0.0000				1.0000
t_2	0.2	0.0000				1.0000
t_1	0.1	0.0000				1.0000
t_0	0.0	0.0000	0.0625	0.2500	0.5625	1.0000
		0.00	0.25	0.50	0.75	1.00
		x_0	x_1	x_2	x_3	x_4

Now let us subsequently compute values at particular time level, using the already computed values from the previous time level:

The first level ($t_1 = 0.1$):

$$U_1^1 = (1 - 2\sigma)\, U_1^0 + \sigma(U_0^0 + U_2^0) + \tau\, f(x_1, t_0) =$$
$$= (1 - 2 \cdot 0.48) \cdot 0.0625 + 0.48(0 + 0.25) + 0.1 \cdot 0.25 = 0.1475$$
$$U_2^1 = (1 - 2\sigma)\, U_2^0 + \sigma(U_1^0 + U_3^0) + \tau\, f(x_2, t_0) =$$
$$= (1 - 2 \cdot 0.48) \cdot 0.25 + 0.48(0.0625 + 0.5625) + 0.1 \cdot 0.50 = 0.36$$
$$U_3^1 = (1 - 2\sigma)\, U_3^0 + \sigma(U_2^0 + U_4^0) + \tau\, f(x_3, t_0) =$$
$$= (1 - 2 \cdot 0.48) \cdot 0.5625 + 0.48(0.25 + 1) + 0.1 \cdot 0.75 = 0.6975$$

The second level ($t_2 = 0.2$):

$$U_1^2 = (1 - 2\sigma)\, U_1^1 + \sigma(U_0^1 + U_2^1) + \tau\, f(x_1, t_1) =$$
$$= (1 - 2 \cdot 0.48) \cdot 0.1475 + 0.48(0 + 0.36) + 0.1 \cdot 0.25 = 0.2037$$
$$U_2^2 = (1 - 2\sigma)\, U_2^1 + \sigma(U_1^1 + U_3^1) + \tau\, f(x_2, t_1) =$$
$$= (1 - 2 \cdot 0.48) \cdot 0.36 + 0.48(0.1475 + 0.6975) + 0.1 \cdot 0.50 = 0.47$$
$$U_3^2 = (1 - 2\sigma)\, U_3^1 + \sigma(U_2^1 + U_4^1) + \tau\, f(x_3, t_1) =$$
$$= (1 - 2 \cdot 0.48) \cdot 0.6975 + 0.48(0.36 + 1) + 0.1 \cdot 0.75 = 0.7557$$

The third and the fourth level ($t = 0.3$ and $t = 0.4$) can be computed similarly. The resulting table with approximate values of the solution at the interior nodes:

t_4	0.4	0.0000	0.2894	0.5846	0.8415	1.0000
t_3	0.3	0.0000	0.2587	0.5293	0.8108	1.0000
t_2	0.2	0.0000	0.2037	0.4700	0.7557	1.0000
t_1	0.1	0.0000	0.1475	0.3600	0.6975	1.0000
t_0	0.0	0.0000	0.0625	0.2500	0.5625	1.0000
		0.00	0.25	0.50	0.75	1.00
		x_0	x_1	x_2	x_3	x_4

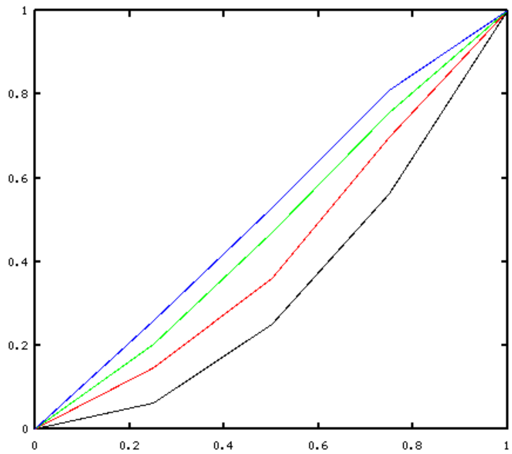

Figure 1: **Problem 1:** Graphs of the solution at time steps from 0 to 0.3 – black, red, green and blue, in succession. The horizontal axis is x, the vertical axis is $u(x,t)$.

b) For $\tau = 0.2$, $h = 0.25$ we have $\sigma = \dfrac{0.3 \cdot 0.2}{0.25^2} = 0.96$. In matrix form, the implicit method can be written as

$$\begin{bmatrix} 1+2\sigma & -\sigma & 0 \\ -\sigma & 1+2\sigma & -\sigma \\ 0 & -\sigma & 1+2\sigma \end{bmatrix} \begin{bmatrix} U_1^{(k+1)} \\ U_2^{(k+1)} \\ U_3^{(k+1)} \end{bmatrix} = \begin{bmatrix} \sigma U_0^{(k+1)} + U_1^{(k)} + \tau f(x_1, t_{k+1}) \\ U_2^{(k)} + \tau f(x_2, t_{k+1}) \\ \sigma U_4^{(k+1)} + U_3^{(k)} + \tau f(x_3, t_{k+1}) \end{bmatrix}$$

The first time level ($t_1 = 0.2$):

$$\begin{bmatrix} 2.92 & -0.96 & 0 \\ -0.96 & 2.92 & -0.96 \\ 0 & -0.96 & 2.92 \end{bmatrix} \begin{bmatrix} U_1^{(1)} \\ U_2^{(1)} \\ U_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 + 0.0625 + 0.2 \cdot 0.25 \\ 0.2500 + 0.2 \cdot 0.50 \\ 0.96 + 0.5625 + 0.2 \cdot 0.75 \end{bmatrix} = \begin{bmatrix} 0.1125 \\ 0.3500 \\ 1.6725 \end{bmatrix}$$

$U^{(1)} = [\ 0.1731,\ 0.4093,\ 0.7074]^T$.

The second time level ($t_2 = 0.4$):

$$\begin{bmatrix} 2.92 & -0.96 & 0 \\ -0.96 & 2.92 & -0.96 \\ 0 & -0.96 & 2.92 \end{bmatrix} \begin{bmatrix} U_1^{(2)} \\ U_2^{(2)} \\ U_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 + 0.1731 + 0.2 \cdot 0.25 \\ 0.4093 + 0.2 \cdot 0.50 \\ 0.96 + 0.7074 + 0.2 \cdot 0.75 \end{bmatrix} = \begin{bmatrix} 0.2231 \\ 0.5093 \\ 1.8174 \end{bmatrix}$$

$U^{(2)} = [\ 0.2459,\ 0.5156,\ 0.7919]^T$.

With the implicit method, the double step-size does not lead to unstability, although the the error is probably about twice as big as in a).

Problem 2

Consider the heat equation

$$\frac{\partial u}{\partial t} = 0.2\,\frac{\partial^2 u}{\partial x^2} + 2t + x \quad \text{in domain } \Omega = (0,1)\times(0,T)$$

with initial condition $u(x,0) = 0$ for $x \in \langle 0,1\rangle$

and boundary conditions $u(0,t) = 0$, $u(1,t) = 3t$ pro $t > 0$.

Choose the spatial and time step-sizes $h = 0.25$ and $\tau = 0.1$, respectively. Verify the stability of the explicit method and compute an approximate value of $u(0.75, 0.4)$.

The solution

$$\sigma = \frac{p\,\tau}{h^2} = \frac{0.2\cdot 0.1}{0.25^2} = 0.32 \le 0.5$$

For the given combination of spatial and time step-sizes, the explicit method is stable.

Let us prepare a table, which then will be subsequently filled by rows from the initial time level, as the time levels are computed one after another. In order to compute the approximate value U_3^4 of $u(0.75, 0.4)$, there is no need to compute all the values inside the domain – the "pyramid" designated in the table will be sufficient:

t_4	0.4				U_3^4	
t_3	0.3			U_2^3	U_3^3	U_4^3
t_2	0.2		U_1^2	U_2^2	U_3^2	U_4^2
t_1	0.1	U_0^1	U_1^1	U_2^1	U_3^1	U_4^1
t_0	0.0	U_0^0	U_1^0	U_2^0	U_3^0	U_4^0
		0.00	0.25	0.50	0.75	1.00
		x_0	x_1	x_2	x_3	x_4

After filling in the initial condition $u(x,0) = 0$ and boundary conditions $u(0,t) = 0$ and $u(1,t) = 3t$ we have

t_4	0.4					
t_3	0.3					0.9000
t_2	0.2					0.6000
t_1	0.1	0.0000				0.3000
t_0	0.0	0.0000	0.0000	0.0000	0.0000	0.0000
		0.00	0.25	0.50	0.75	1.00
		x_0	x_1	x_2	x_3	x_4

Now let us subsequently compute values at particular time levels:

The first level ($t_1 = 0.1$):

$$U_1^1 = (1-2\sigma)\,U_1^0 + \sigma(U_0^0 + U_2^0) + \tau\, f(x_1, t_0) =$$
$$= (1 - 2\cdot 0.32)\cdot 0 + 0.32(0+0) + 0.1(2\cdot 0 + 0.25) = 0.025$$

$$U_2^1 = (1-2\sigma)\,U_2^0 + \sigma(U_1^0 + U_3^0) + \tau\, f(x_2, t_0) =$$
$$= 0.36\cdot 0 + 0.32(0+0) + 0.1(2\cdot 0 + 0.5) = 0.05$$

$$U_3^1 = (1-2\sigma)\,U_3^0 + \sigma(U_2^0 + U_4^0) + \tau\, f(x_3, t_0) =$$
$$= 0.36\cdot 0 + 0.32(0+0) + 0.1(2\cdot 0 + 0.75) = 0.075$$

The second level ($t_2 = 0.2$):

$$U_1^2 = (1-2\sigma)\,U_1^1 + \sigma(U_0^1 + U_2^1) + \tau\, f(x_1, t_1) =$$
$$= 0.36\cdot 0.025 + 0.32(0 + 0.05) + 0.1(2\cdot 0.1 + 0.25) = 0.07$$

$$U_2^2 = (1-2\sigma)\,U_2^1 + \sigma(U_1^1 + U_3^1) + \tau\, f(x_2, t_1) =$$
$$= 0.36\cdot 0.05 + 0.32(0.025 + 0.075) + 0.1(2\cdot 0.1 + 0.5) = 0.12$$

$$U_3^2 = (1-2\sigma)\,U_3^1 + \sigma(U_2^1 + U_4^1) + \tau\, f(x_3, t_1) =$$
$$= 0.36\cdot 0.075 + 0.32(0.05 + 0.3) + 0.1(2\cdot 0.1 + 0.75) = 0.234$$

Te third level ($t_3 = 0.3$):

$$U_2^3 = (1-2\sigma)\,U_2^2 + \sigma(U_1^2 + U_3^2) + \tau\, f(x_2, t_2) =$$
$$= 0.36\cdot 0.12 + 0.32(0.07 + 0.234) + 0.1(2\cdot 0.2 + 0.5) = 0.2305$$

$$U_3^3 = (1-2\sigma)\,U_3^2 + \sigma(U_2^2 + U_4^2) + \tau\, f(x_3, t_2) =$$
$$= 0.36\cdot 0.234 + 0.32(0.12 + 0.6) + 0.1(2\cdot 0.2 + 0.75) = 0.4296$$

The fourth level ($t_4 = 0.4$):

$$U_3^4 = (1-2\sigma)\,U_3^3 + \sigma(U_2^3 + U_4^3) + \tau\, f(x_3, t_3) =$$
$$= 0.36\cdot 0.4296 + 0.32(0.2305 + 0.9) + 0.1(2\cdot 0.3 + 0.75) = 0.6514$$

The resulting table with approximate values of the solution:

t_4	0.4				0.6514	
t_3	0.3			0.2305	0.4296	0.9000
t_2	0.2		0.0700	0.1200	0.2340	0.6000
t_1	0.1	0.0000	0.0250	0.0500	0.0750	0.3000
t_0	0.0	0.0000	0.0000	0.0000	0.0000	0.0000
		0.00	0.25	0.50	0.75	1.00
		x_0	x_1	x_2	x_3	x_4

The approximate value of $u(0.75, 0.4)$ is equal to 0.6514 .