Partial differential equations I

The theory (very short excerpts from the lectures)

Second order linear PDE in 2 variables

We want to find a function $u \equiv u(x, y)$, which satisfies the equation

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + hu = f \qquad (1)$$

in a given domain Ω, where $a \equiv a(x,y), b \equiv b(x,y), \dots h \equiv h(x,y)$ and $f \equiv f(x,y)$ are functions continuous on Ω. Moreover, we demand that $u(x,y)$ fulfills some given conditions on the boundary Γ of the domain Ω.

Classification of the 2-nd order linear PDE in 2 variables

The mathematical nature of the solutions of the equation (1) depends on the algebraic properties of the polynomial $ax^2 + bxy + cy^2 + dx + ey + q$; numerical method for solving the equation should be chosen accordingly to the type of the equation. The equations are classified by the sign of the discriminant $r(x,y) = (b(x,y))^2 - 4\,a(x,y)\,c(x,y)$. There are three types of equations:

- elliptic $\dots r(x,y) < 0$ (for example Poisson's equation)
- parabolic $\dots r(x,y) = 0$ (for example heat equation)
- hyperbolic $\dots r(x,y) > 0$ (for example wave equation)

Dirichlet problem for Poisson equation - finite difference method

We are seeking a function $u \equiv u(x,y)$ which satisfies

$$-\Delta u \;=\; f(x,y)\,, \qquad \text{where} \quad \Delta u \;\equiv\; \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\,,$$

in the domain Ω and has prescribed values $u(x,y) = \phi(x,y)$ on its boundary Γ.

- Choose a step-size h and cover the domain Ω by a square mesh.
- Denote by $P_i = [x_i, y_i]$, $i = 1, \dots n$, the nodes inside Ω and by U_i the approximate value of the solution u in P_i: $U_i \approx u(x_i, y_i) \equiv u(P_i)$.
- In every interior node P_i, assemble one equation which ties together the value U_i with values U_H, U_D, U_L and U_P in adjacent nodes P_H, P_D, P_L and P_P (upper, lower, left and right neighbor), respectively:

  $4U_i - U_H - U_D - U_L - U_P = h^2\, f(x_i, y_i)$, $i = 1, \dots n$

  If some neighbor lies on Γ, use corresponding prescribed boundary value and substitute it to the equation (if some neighbor lies outside Ω, use linear interpolation instead).
- Approximate values $U_1, \dots U_n$ of the solution in the interior nodes $P_1, \dots P_n$ are obtained by solving the system of n equations for these n variables.

Problem 1

Consider the equation

$$x^2y^2\frac{\partial^2 u}{\partial x^2} - xy\frac{\partial^2 u}{\partial x \partial y} + 0.25\frac{\partial^2 u}{\partial y^2} = x + 2y$$

How is this equation classified?

The solution

We are interested in the sign of the determinant
$r(x,y) = (b(x,y))^2 - 4\,a(x,y)\,c(x,y) = (-xy)^2 - 4\,x^2y^2\;\cdot 0.25 = 0$.
Function $r(x,y)$ is equal to zero for all x, y, so the given equation is classified as parabolic (in any domain).

Problem 2

Consider Dirichlet problem for Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = y - x \quad \text{in } \Omega\,, \quad u(x,y) = y \quad \text{on } \Gamma$$

where Ω is a pentagon with vertices [− 1, 0], [− 1, 1.5], [0, 1.5], [0.5, 1] and [− 0.5, 0].

Choose the step-size $h = 0.5$ and compute approximate solution at the point [− 0.5, 1] using finite difference method.

The solution

In order to compute approximate value at the given point, we have to design a mesh over the domain, so that the point is one of the nodes of the mesh, and compute values at all nodes of the mesh. First of all, let us sketch the picture of the domain and the mesh and denote the nodes which we will use:

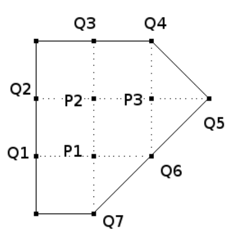

Figure 1: The given domain with interior nodes denoted as P_i and boundary nodes denoted as Q_j .

There are 3 interior nodes $P_1 = [\,-0.5, 0.5\,]$, $P_2 = [\,-0.5, 1\,]$ and $P_3 = [\,0, 1\,]$ and 9 boundary nodes, from which we will use just 7 denoted as Q_j. We want to compute approximate values U_1, U_2 and U_3 of the solution at nodes P_i. Values at the nodes Q_j can be computed in advance from the boundary conditions:

$u(Q_1) = u(\,-1\,,\ 0.5) = 0.5$
$u(Q_2) = u(\,-1\,,\ 1) = 1$
$u(Q_3) = u(\,-0.5\,,\ 1.5) = 1.5$
$u(Q_4) = u(\,0\,,\ 1.5) = 1.5$
$u(Q_5) = u(\,0.5\,,\ 1) = 1$
$u(Q_6) = u(\,0\,,\ 0.5) = 0.5$
$u(Q_7) = u(\,-0.5\,,\ 0) = 0$

Let us prepare also the values of $f(P_i)$:

$f(P_1) = y_1 - x_1 = 0.5 - (-0.5) = 1$
$f(P_2) = y_2 - x_2 = 1 - (-0.5) = 1.5$
$f(P_3) = y_3 - x_3 = 1 - 0 = 1$

Now we can assemble (and rearrange) one equation at every node P_i:

P_1:

$4U_1 - U_2 - u(Q_1) - u(Q_6) - u(Q_7) = -h^2\, f(P_1)$
$4U_1 - U_2 = -h^2\, f(P_1) + u(Q_1) + u(Q_6) + u(Q_7)$
$4U_1 - U_2 = -0.25 \cdot 1 + 0.5 + 0.5 + 0 = 0.75$
$4U_1 - U_2 = 0.75$

P_2:

$4U_2 - U_1 - U_3 - u(Q_2) - u(Q_3) = -h^2\, f(P_2)$
$4U_2 - U_1 - U_3 = -h^2\, f(P_2) + u(Q_2) + u(Q_3)$
$4U_2 - U_1 - U_3 = -0.25 \cdot 1.5 + 1 + 1.5 = 2.125$
$4U_2 - U_1 - U_3 = 2.125$

P_3:

$4U_3 - U_2 - u(Q_4) - u(Q_5) - u(Q_6) = -h^2\, f(P_3)$
$4U_3 - U_2 = -h^2\, f(P_3) + u(Q_4) + u(Q_5) + u(Q_6)$
$4U_3 - U_2 = -0.25 \cdot 1 + 1.5 + 1 + 0.5 = 2.75$
$4U_3 - U_2 = 2.75$

The resulting system of linear equations is

$$\begin{aligned} 4U_1 - U_2 \quad\quad &= 0.75 \\ 4U_2 - U_1 - U_3 &= 2.125 \\ 4U_3 - U_2 \quad\quad &= 2.75 \end{aligned}$$

In matrix form

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 2.125 \\ 2.75 \end{bmatrix}$$

The solution: $U_1 = 0.4018,\ U_2 = 0.8571,\ U_3 = 0.9018$.

The approximate solution at the point $P_2 = [\,-0.5, 1\,]$ is $U_2 = 0.8571$.

Problem 3

Consider the same problem as in Problem 2 with a small change of the domain Ω: move its last vertex a little to the right, so that the domain is now given by the vertices [− 1, 0], [− 1, 1.5], [0, 1.5], [0.5, 1] and **[− 0.3, 0]**.

The solution

There are four interior nodes now: the *regular* nodes P_1, P_2, P_3 (regular node has no neighbor outside $\bar{\Omega}$) and the new node $P_4 = [\ 0,\ 0.5\ ]$, which was originally the boundary node Q_6 . The node P_4 is *irregular* in the sense that its right mesh neighbor does not lie in the domain nor at the boundary (sketch a picture of it).

At regular nodes, the equations remain the same (with the only exception: there is a new unknown U_4 instead of the given boundary value $u(Q_6)$):

P_1:

$$4U_1 - U_2 - U_4 - u(Q_1) - u(Q_7) = -h^2\, f(P_1)$$
$$4U_1 - U_2 - U_4 = -h^2\, f(P_1) + u(Q_1) + u(Q_7)$$
$$4U_1 - U_2 - U_4 = -0.25 \cdot 1 + 0.5 + 0 = 0.25$$
$$4U_1 - U_2 - U_4 = 0.25$$

P_2:

$$4U_2 - U_1 - U_3 - u(Q_2) - u(Q_3) = -h^2\, f(P_2)$$
$$\cdots$$
$$4U_2 - U_1 - U_3 = 2.125$$

P_3:

$$4U_3 - U_2 - U_4 - u(Q_4) - u(Q_5) = -h^2\, f(P_3)$$
$$4U_3 - U_2 - U_4 = -h^2\, f(P_3) + u(Q_4) + u(Q_5)$$
$$4U_3 - U_2 - U_4 = -0.25 \cdot 1 + 1.5 + 1 = 2.25$$
$$4U_3 - U_2 - U_4 = 2.25$$

The fourth equation is given by linear interpolation of the value at P_4 from the values at P_1 and at the auxiliary point Q_8 – the intersection of the line P_1P_4 and the boundary. From similarity of triangles we obtain $Q_8 = [\ 0.1,\ 0.5\ ]$ with the prescribed value of $u(Q_8) = u(\ 0.1\ ,\ 0.5) = 0.5$ and from linear interpolation:

$$(U_4 - U_1)/h \quad = (u(Q_8) - U_1)/(h + dist(P_4, Q_8))$$
$$(U_4 - U_1)/0.5 \ = (0.5 - U_1)/0.6$$
$$0.6\, U_4 - 0.1\, U_1 = 0.25$$

The matrix form of the equations is

$$\begin{bmatrix} 4 & -1 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ -0.1 & 0 & 0 & 0.6 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 2.125 \\ 2.25 \\ 0.25 \end{bmatrix}$$

and the solution: $U_1 = 0.3969,\ U_2 = 0.8547,\ U_3 = 0.8969,\ U_4 = 0.4828$.