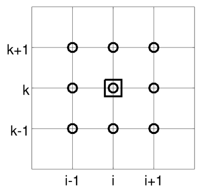
Finite differences for second order linear PDE in 2 variables

Approximation of partial derivatives with finite differences

Consider a PDE with the state variables x and t solved in a domain Ω.
Let us construct a rectangular grid of nodes over Ω with equal mesh spacing h in the x-direction and equal mesh spacing τ in the t-direction.

Scheme of the grid around a grid node P_i^k:



Notation:

$P_i^k \equiv [x_i, t_k] \ldots$ grid nodes, where

$x_i \ldots x$-coordinates of the nodes: $h = x_{i+1} - x_i$

$t_k \ldots t$-coordinates of the nodes: $\tau = t_{k+1} - t_k$

$u(x,t) \ldots$ function of two variables defined in Ω, $u(P_i^k) \equiv u(x_i, t_k)$

$U_i^k \approx u(P_i^k) \ldots$ approximate value of $u(x,t)$ at a grid node P_i^k

Partial derivatives then can be approximated as (see Figure 1)

$$\frac{\partial^2 u}{\partial x^2}(P_i^k) \approx \frac{U_{i-1}^k - 2U_i^k + U_{i+1}^k}{h^2} \tag{1}$$

$$\frac{\partial^2 u}{\partial t^2}(P_i^k) \approx \frac{U_i^{k-1} - 2U_i^k + U_i^{k+1}}{\tau^2} \tag{2}$$

$$\frac{\partial u}{\partial t}(P_i^k) \approx \frac{U_i^{k+1} - U_i^k}{\tau} \tag{3}$$

$$\frac{\partial u}{\partial t}(P_i^k) \approx \frac{U_i^k - U_i^{k-1}}{\tau} \tag{4}$$

where
(1) the second central difference with respect to x,
(2) the second central difference with respect to t,
(3) the first forward difference with respect to t, and
(4) the first backward difference with respect to t
were used for approximation of the derivatives at the node P_i^k.

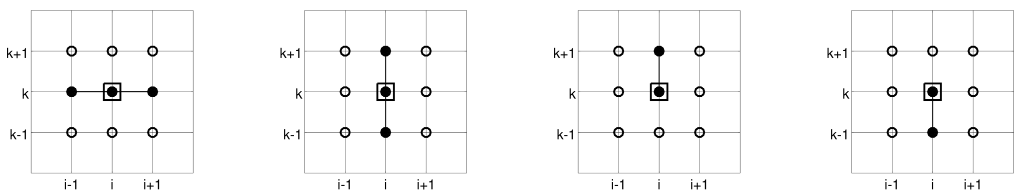


Figure 1: Grid nodes used for finite differences centered at the node P_i^k. Cases (1), (2), (3) and (4) above, from left to right.

Discretization of partial differential equations in 2D using finite differences

Discretization of PDE (inside the given domain) consists of the three following steps:

1. Choosing the step-size in both directions and constructing the grid.
2. Expressing the equation at every grid node (inside the domain).
3. Substitution of derivatives with the finite differences.

Caution: All terms of the equation have to be expressed or approximated at the same position.

Note: Discretization of initial and boundary conditions is not covered in this text.

Examples

1. Poisson equation

$$-\Delta u \;=\; f(x,y)\,, \qquad \text{where} \quad \Delta u \;\equiv\; \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\;,$$

where both x and y are treated the same way (usually represent spatial directions).

- Choose the same step-size h in both directions x and y.
- Express the equation at every interior node $P_i^k = [x_i, y_k]$:

$$-\frac{\partial^2 u}{\partial x^2}(P_i^k) - \frac{\partial^2 u}{\partial y^2}(P_i^k) \;=\; f(P_i^k)$$

- Use five-point stencil for discretization (see Figure 2), which represents substitution of second derivatives with second central finite differences, using formulae (1) and (2):

$$-\frac{U_{i-1}^k - 2U_i^k + U_{i+1}^k}{h^2} - \frac{U_i^{k-1} - 2U_i^k + U_i^{k+1}}{h^2} \;=\; f(x_i, y_k)\;.$$

  After rearrangig this leads to equation for 5 unknowns:

$$4U_i^k - U_{i-1}^k - U_{i+1}^k - U_i^{k-1} - U_i^{k+1} \;=\; h^2\, f(x_i, y_k)\;.$$

The discretization is performed at every inner node, then a system of linear equations for unknowns U_i^k is obtained.

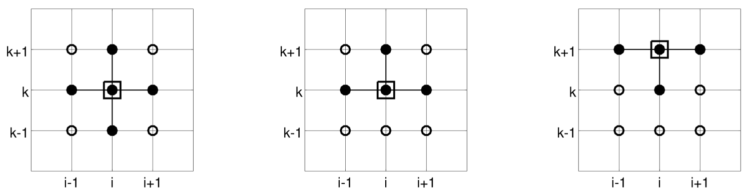


Figure 2: Discretization schemas, from left to right: five-point stencil, four-point stencil leading to explicit method, four-point stencil leading to implicit method. The node, where the discretization takes place, is indicated by a square.

2. Heat transfer equation

$$\frac{\partial u}{\partial t} = p\,\frac{\partial^2 u}{\partial x^2} + f(x,t) \quad \text{in the domain } \Omega = (a,b)\times(0,T)\ ,$$

where variable t represents time and x is spatial variable.

As x and t now represent different entities, different step-sizes h and τ are used.

Numerical solution is evaluated one time level after another: from known values at the k-th time level, values at $(k+1)$-st time level are computed. (Assume that the starting, zero level is prescribed by an initial condition.)

Explicit method

- Express the equation at the node $P_i^k = [x_i, t_k]$:

$$\frac{\partial u}{\partial t}(P_i^k) = p\,\frac{\partial^2 u}{\partial x^2}(P_i^k) + f(x_i, t_k)$$

- For discretization, use four-point stencil depicted in Figure 2, center, which represents substitution of time derivative with first forward finite difference by formula (3) and space derivative with second central difference by formula (1):

$$\frac{U_i^{k+1} - U_i^k}{\tau} \;=\; p\,\frac{U_{i-1}^k - 2U_i^k + U_{i+1}^k}{h^2} \;+\; f(x_i, t_k)\ .$$

  After rearrangig, the explicit formula is obtained:

$$U_i^{k+1} = \sigma\,U_{i-1}^k + (1-2\sigma)\,U_i^k + \sigma\,U_{i+1}^k \;+\; \tau\,f(x_i,t_k)\ , \quad \text{where } \sigma = \frac{p\,\tau}{h^2}$$

Computation of the $(k+1)$-st time level is performed one value after another.

Implicit method

- Express the equation at the node $P_i^{k+1} = [x_i, t_{k+1}]$:

$$\frac{\partial u}{\partial t}(P_i^{k+1}) = p\,\frac{\partial^2 u}{\partial x^2}(P_i^{k+1}) + f(x_i, t_{k+1})$$

- For discretization, use four-point stencil depicted in Figure 2, right, which represents substitution of time derivative with first backward finite difference by formula (4) and space derivative with second central difference by formula (1), both expressed at node P_i^{k+1}:

$$\frac{U_i^{k+1} - U_i^k}{\tau} \;=\; p\,\frac{U_{i-1}^{k+1} - 2U_i^{k+1} + U_{i+1}^{k+1}}{h^2} \;+\; f(x_i, t_{k+1})\ .$$

  After rearrangig, this leads to equation for 3 unknowns at the $(k+1)$-st time level:

$$-\sigma\,U_{i-1}^{k+1} + (1+2\sigma)\,U_i^{k+1} \;-\sigma\,U_{i+1}^{k+1} = U_i^k \;+\; \tau\,f(x_i,t_{k+1})\ , \quad \text{where } \sigma \;=\; \frac{p\,\tau}{h^2}$$

The discretization is performed at every inner node of the $(k+1)$-st time level, so a system of linear equations is obtained, from which values at the $(k+1)$-st time level can be computed.

3. Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2} + f(x,t) \quad \text{in the domain } \Omega = (a,b)\times(0,T)\ ,$$

where variable t represents time and x is spatial variable.

As x and t represent different entities, different step-sizes h and τ are used.

Numerical solution is evaluated one time level after another: from known values at $(k-1)$-st and k-th time level, values at $(k+1)$-st time level are computed. (Assume that the starting two levels (zero and first) are computed from initial conditions.)

Explicit method

- Express the equation at the node $P_i^k = [x_i, t_k]$:

$$\frac{\partial^2 u}{\partial t^2}(P_i^k) = c^2\,\frac{\partial^2 u}{\partial x^2}(P_i^k) + f(x_i, t_k)$$

- Use five-point stencil for discretization (see Figure 2) using formulae (1) and (2)::

$$\frac{U_i^{k-1} - 2U_i^k + U_i^{k+1}}{\tau^2} \;=\; c^2\,\frac{U_{i-1}^k - 2U_i^k + U_{i+1}^k}{h^2} \;+\; f(x_i, t_k)\ .$$

After rearrangig, the explicit formula is obtained:

$$U_i^{k+1} = \sigma^2\,U_{i-1}^k + 2\,(1-\sigma^2)\,U_i^k + \sigma^2\,U_{i+1}^k \;-\; U_i^{k-1} \;+\; \tau^2\,f(x_i,t_k)\ , \quad \text{where } \sigma = \frac{c\,\tau}{h}$$

Computation of the $(k+1)$-st time level is performed one value after another.