ODE - initial value (or Cauchy) problems

A first-order ordinary differential equation, initial value problem

$$y' = f(x, y) \quad \text{with the initial condition} \;\; y(x^{(0)}) = y^{(0)} \,. \tag{1}$$

Existence and uniqueness of the (exact) solution

Suppose that the function $f(x, y)$ and its partial derivative $\dfrac{\partial f}{\partial y}$ are continuous in a domain $\Omega \subset R^2$. Then every point $[x^{(0)}, y^{(0)}] \in \Omega$ determines a unique maximal solution $y(x)$ of (1), $[x, y(x)] \subset \Omega$.

The equation (1) is called *linear*, if the function f is linear with respect to variable y , i.e. it has a form $f(x, y) = g_0(x) + g_1(x)\, y$. For a linear equation the following holds:

Suppose that the functions $g_0(x)$ and $g_1(x)$ are continuous on an interval I. Then every point $[x^{(0)}, y^{(0)}] \in I \times R$ determines a unique maximal solution of (1), defined on the whole interval I.

Explicit Euler method (or Euler **forward** method)

choose a step size h and for $k = 0, 1, 2, \ldots$ compute

$x^{(k+1)} = x^{(k)} + h$

$y^{(k+1)} = y^{(k)} + h\, f(x^{(k)}, y^{(k)})$

Where this formula comes from:

$$\begin{aligned}
y'(x^{(k)}) &= f(x^{(k)}, y(x^{(k)})) \\
\frac{y(x^{(k+1)}) - y(x^{(k)})}{h} + \mathcal{O}(h) &= f(x^{(k)}, y(x^{(k)})) \\
y(x^{(k+1)}) &= y(x^{(k)}) + h\, f(x^{(k)}, y(x^{(k)})) + \mathcal{O}(h^2) \\
y^{(k+1)} &= y^{(k)} + h\, f(x^{(k)}, y^{(k)})
\end{aligned}$$

Implicit Euler method (or Euler **backward** method)

choose a step size h and for $k = 0, 1, 2, \ldots$ compute

$x^{(k+1)} = x^{(k)} + h$

$y^{(k+1)} = y^{(k)} + h\, f(x^{(k+1)}, y^{(k+1)})$

this is an implicit equation for $y^{(k+1)}$, it can be solved using fixed point iterations or Newton's method.

Inference:

$$\begin{aligned}
y'(x^{(k+1)}) &= f(x^{(k+1)}, y(x^{(k+1)})) \\
\frac{y(x^{(k+1)}) - y(x^{(k)})}{h} + \mathcal{O}(h) &= f(x^{(k+1)}, y(x^{(k+1)})) \\
y(x^{(k+1)}) &= y(x^{(k)}) + h\, f(x^{(k+1)}, y(x^{(k+1)})) + \mathcal{O}(h^2) \\
y^{(k+1)} &= y^{(k)} + h\, f(x^{(k+1)}, y^{(k+1)})
\end{aligned}$$

Example 1

Consider Cauchy problem $y' = \dfrac{y}{x^2}\,, \quad y(1) = 2\,.$

1) Find a domain where the existence of a unique solution of the problem is guaranteed.

2) Compute an approximate value of $y(1.4)$ using:
   a) Explicit Euler method with step size $h = 0.2$,
   b) Implicit Euler method with step size $h = 0.2$,
   c) Explicit and Implicit Euler method with step size $h = 0.1$.

The solution

1) This is linear differential equation with coefficients $g_0(x) = 0$ and $g_1(x) = \frac{1}{x^2}$, continuous in the intervals $I_1 = (-\infty, 0)$ and $I_2 = (0, \infty)$. As $x^{(0)} = 1$ lies in I_2, the interval of maximal solution of the given problem is I_2.

2) The results are summarized in Table 1 and for explicit Euler method also depicted in Figure 1.

Computation:

a) $h = 0.2, \quad x^{(0)} = 1, \quad y^{(0)} = 2$

$k \equiv f(x^{(0)}, y^{(0)}) = \dfrac{y^{(0)}}{(x^{(0)})^2} = \dfrac{2}{1^2} = 2$

$x^{(1)} = x^{(0)} + h = 1 + 0.2 = 1.2, \quad y^{(1)} = y^{(0)} + h\,k = 2 + 0.2 \cdot 2 = 2.4$

$k \equiv f(x^{(1)}, y^{(1)}) = \dfrac{y^{(1)}}{(x^{(1)})^2} = \dfrac{2.4}{(1.2)^2} = 1.6667$

$x^{(2)} = x^{(1)} + h = 1.2 + 0.2 = 1.4, \quad y^{(2)} = y^{(1)} + h\,k = 2.4 + 0.2 \cdot 1.6667 = 2.7333$

$y(1.4)$ is approximately equal to $y^{(2)} = 2.7333$.

b) There is no general explicit formula: in every iteration, we have to solve the equation $y^{(k+1)} = y^{(k)} + h\, f(x^{(k+1)}, y^{(k+1)})$; for this problem it is

$$y^{(k+1)} = y^{(k)} + h\,\frac{y^{(k+1)}}{(x^{(k+1)})^2}\;.$$

In the case of *linear* differential equation like this, however, we can express $y^{(k+1)}$ from the equation above explicitly:

$$y^{(k+1)} = \frac{(x^{(k+1)})^2}{(x^{(k+1)})^2 - h}\; y^{(k)}\;.$$

$h = 0.2, \quad x^{(0)} = 1, \quad y^{(0)} = 2$

$x^{(1)} = x^{(0)} + h = 1 + 0.2 = 1.2\;, \quad y^{(1)} = \frac{(x^{(1)})^2}{(x^{(1)})^2 - h}\; y^{(0)} = \frac{1.2^2}{1.2^2 - 0.2} \cdot 2 = 2.3226$

$x^{(2)} = x^{(1)} + h = 1.2 + 0.2 = 1.4\;, \quad y^{(2)} = \frac{(x^{(2)})^2}{(x^{(2)})^2 - h}\; y^{(1)} = \frac{1.4^2}{1.4^2 - 0.2} \cdot 2.3226 = 2.5865$

$y(1.4)$ is approximately equal to $y^{(2)} = 2.5865$.

c) Using similar process as in a), we obtain values presented at the second column of Table 1:

$y(1.4)$ is approximately equal to $y^{(4)} = 2.6979$ for explicit Euler method and to $y^{(4)} = 2.6241$ for the implicit one.

$x^{(i)}$	exact $y(x^{(i)})$	Euler Explic.	$h = 0.1$ Implic.	Euler Explic.	$h = 0.2$ Implic.
1	2.0000	2.0000	2.0000	2.0000	2.0000
1.1	2.1903	2.2000	2.1802		
1.2	2.3627	2.3818	2.3429	2.4000	2.3226
1.3	2.5191	2.5472	2.4902		
1.4	2.6614	2.6979	2.6241	2.7333	2.5865
1.5	2.7912	2.8356	2.7462		
1.6	2.9100	2.9616	2.8578	3.0122	2.8057
1.7	3.0190	3.0773	2.9602		
1.8	3.1192	3.1838	3.0545	3.2476	2.9903
1.9	3.2118	3.2821	3.1415		
2.0	3.2974	3.3730	3.2221	3.4480	3.1477

Table 1: **Example 1**. The first column represents values of x where the approximate solution is computed. At the second column there is exact solution, the third column presents approximate solutions obtained by Euler methods with step size $h = 0.1$ and in the last column there are results for the step size $h = 0.2$. Results for explicit Euler method are depicted in Figure 1.

Observations and questions

A) The errors of both explicit and implicit Euler methods are very similar. From the last row in the Table 1, the errors of explicit and implicit Euler methods can be computed: they are 0.0756 and 0.0753 for step $h = 0.1$ and 0.1506 and 0.1497 for step $h = 0.2$.

Why should then we bother using the implicit method, which generally requires to solve an implicit equation in every step? Because of *stability* (will be covered in next lectures).

B) Using half step size, the errors are reduced approximately twice. Does this hold for any numerical method of solving ODE? Could it be improved? The speed of *convergence* will be discussed in next lectures.

A system of first-order ODEs, initial value problem

$$\mathbf{Y}'(x) = \mathbf{F}(x, \mathbf{Y}(x)) \quad \text{with an initial condition } \mathbf{Y}(x^{(0)}) = \mathbf{Y}^{(0)}, \tag{2}$$

where

$$\mathbf{Y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}, \; \mathbf{Y}'(x) = \begin{bmatrix} y_1'(x) \\ y_2'(x) \\ \vdots \\ y_n'(x) \end{bmatrix}, \; \mathbf{F}(x, \mathbf{Y}) = \begin{bmatrix} f_1(x, y_1, y_2, \ldots, y_n) \\ f_2(x, y_1, y_2, \ldots, y_n) \\ \vdots \\ f_n(x, y_1, y_2, \ldots, y_n) \end{bmatrix}$$

Existence and uniqueness of the (exact) solution

Let functions f_i be continuous in a domain $\Omega \subset R^{n+1}$ and have also continuous partial derivatives $\dfrac{\partial f_i}{\partial y_j}$, $i, j = 1, \ldots, n$ there. Then every point $[x^{(0)}, \mathbf{Y}^{(0)}] \in \Omega$ determines a unique maximal solution $\mathbf{Y}(x)$ of (2) and $[x, \mathbf{Y}(x)] \subset \Omega$.

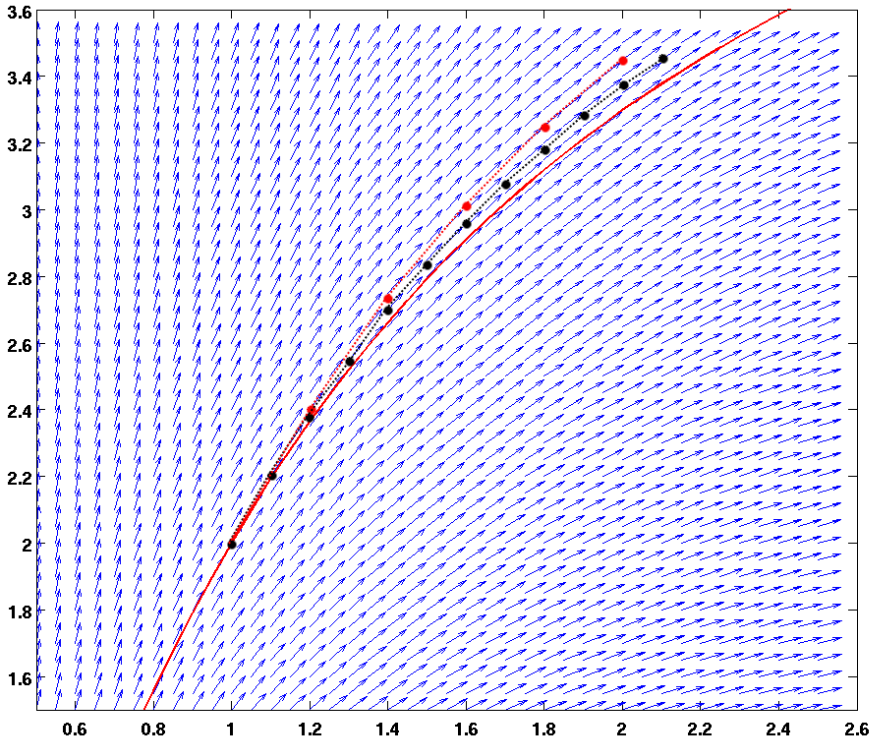

Figure 1: **Example 1**. Horizontal axis is x, vertical axis is y. Every blue arrow represent tangent vector to the integral curve passing through the matching point. The full red line represents the exact solution of the problem with a given initial condition $y(x) = 2\,e^{1-1/x}$ (it can be computed using a separation of variables). Red and black points represent approximation of the solution computed by Euler method with step size 0.2 and 0.1, respectively (see also Table 1).

The system (2) is called *linear*, if all functions f_i are linear with respect to all variables y_j , i.e. they have the form $f_i(x, y_1, y_2, \dots, y_n) = g_{i0}(x) + g_{i1}(x)\,y_1 + g_{i2}(x)\,y_2 + ... + g_{in}(x)\,y_n$. Linear system can be written in matrix form as

$\mathbf{Y}' = \mathbf{G}\,\mathbf{Y} + \mathbf{G}_0$, where $\mathbf{G} = \{g_{ij}\}_{i,j=1}^{n}$ and $\mathbf{G}_0 = (g_{1\,0}, g_{2\,0}, \dots g_{n0})^T$,

and the following holds:

Let all functions $g_{i0}(x)$, $g_{ij}(x)$ be continuous on an interval I. Then, for every $x^{(0)} \in I$ and $\mathbf{Y}^{(0)} \in R^n$, the point $[x^{(0)}, \mathbf{Y}^{(0)}]$ determines a unique maximal solution $\mathbf{Y}(x)$ of (2) defined on the whole interval I.

Explicit Euler method

choose a step size h and for $k = 0, 1, 2, \dots$

1. compute the derivative $\mathbf{K}$ of the vector function $\mathbf{Y}$ as $\mathbf{K} = \mathbf{F}(x^{(k)}, \mathbf{Y}^{(k)})$
2. set

$$x^{(k+1)} = x^{(k)} + h$$
$$\mathbf{Y}^{(k+1)} = \mathbf{Y}^{(k)} + h\,\mathbf{K}$$

Implicit Euler method

choose a step size h and for $k = 0, 1, 2, \dots$

1. set $x^{(k+1)} = x^{(k)} + h$
2. compute $\mathbf{Y}^{(k+1)}$ from the equation (using fixed point iterations or Newton's method)
   $\mathbf{Y}^{(k+1)} = \mathbf{Y}^{(k)} + h\,\mathbf{F}(x^{(k+1)},\, \mathbf{Y}^{(k+1)})$

For linear system $\mathbf{Y}' = \mathbf{G}\,\mathbf{Y} + \mathbf{G}_0$, the equation above represents a linear system
$(\mathbf{E} - h\,\mathbf{G})\,\mathbf{Y}^{(k+1)} = \mathbf{Y}^{(k)} + h\,\mathbf{G}_0$ (both $\mathbf{G}$ and $\mathbf{G}_0$ generally depend on $x^{(k+1)}$).

Example 2

Consider Cauchy problem

$$\mathbf{Y}' = \begin{bmatrix} y_1 \sin(x) + y_3 \\ y_2 \ln(x+1) - 4 \\ 2y_1 - \dfrac{y_3}{x-2} \end{bmatrix}, \qquad \mathbf{Y}(1) = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

a) Check that the given problem has unique solution and find the interval I of its maximal solution.

b) Choose a step size $h = 0.1$ and compute an approximate value of $\mathbf{Y}(1.2)$ using Euler (explicit) method.

The solution

a) The system of equations is linear, so continuity of its coefficients has to be checked only:
$x + 1 > 0 \;\Rightarrow\; x > -1, \quad x - 2 \neq 0 \;\Rightarrow\; x \neq 2 \qquad I_1 = (-1, 2),\; I_2 = (2, \infty)$
$x^{(0)} = 1 \in I_1 \Rightarrow$ interval of maximal solution is $(-1, 2)$.

b) $x^{(0)} = 1$, $\mathbf{Y}^{(0)} = (-1, 1, 2)^T$, $h = 0.1$:

$$\mathbf{K} = \mathbf{F}(x^{(0)}, \mathbf{Y}^{(0)}) = \begin{bmatrix} -1 \cdot \sin(1) + 2 \\ 1 \cdot \ln(1+1) - 4 \\ 2 \cdot (-1) - \dfrac{2}{1-2} \end{bmatrix} = \begin{bmatrix} -0.84147 + 2 \\ 0.69315 - 4 \\ -2 + 2 \end{bmatrix} = \begin{bmatrix} 1.1585 \\ -3.3068 \\ 0 \end{bmatrix}$$

$x^{(1)} = x^{(0)} + h = 1 + 0.1 = 1.1$

$$\mathbf{Y}^{(1)} = \mathbf{Y}^{(0)} + h\,\mathbf{K} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + 0.1 \begin{bmatrix} 1.1585 \\ -3.3068 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.8842 \\ 0.6693 \\ 2 \end{bmatrix}$$

$$\mathbf{K} = \mathbf{F}(x^{(1)}, \mathbf{Y}^{(1)}) = \begin{bmatrix} -0.8842 \cdot \sin(1.1) + 2 \\ 0.6693 \cdot \ln(1.1+1) - 4 \\ 2 \cdot (-0.8842) - \dfrac{2}{1.1-2} \end{bmatrix} = \begin{bmatrix} -0.7880 + 2 \\ 0.6693 \cdot 0.74194 - 4 \\ -1.7684 + 2.2222 \end{bmatrix} = \begin{bmatrix} 1.2120 \\ -3.5034 \\ 0.45380 \end{bmatrix}$$

$x^{(2)} = x^{(1)} + h = 1.1 + 0.1 = 1.2$

$$\mathbf{Y}^{(2)} = \mathbf{Y}^{(1)} + h\,\mathbf{K} = \begin{bmatrix} -0.8842 \\ 0.6693 \\ 2 \end{bmatrix} + 0.1 \begin{bmatrix} 1.2120 \\ -3.5034 \\ 0.4538 \end{bmatrix} = \begin{bmatrix} -0.7630 \\ 0.3190 \\ 2.0454 \end{bmatrix}$$

The value of $\mathbf{Y}(1.2)$ is approximately $\mathbf{Y}^{(2)} = (-0.7630, 0.3190, 2.0454)^T$.

Higher-order initial value problems

differential equation of the n-th order:

$$y^n(x) = f(x, y, y', y'', \dots y^{n-1}) \quad \text{with initial conditions}$$
$$y(x^{(0)}) = y_1^{(0)}, \quad y'(x^{(0)}) = y_2^{(0)}, \quad \dots \quad y^{n-1}(x^{(0)}) = y_n^{(0)} \tag{3}$$

In order to be able to use Euler methods, we need to represent this diferential equation of n-th order as n first-order diferential equations. Introducing auxiliary variables $y_1 = y$, $y_2 = y'$, $y_3 = y''$, $\dots y_n = y^{n-1}$ into equation (3) leads to a system

$$\mathbf{F}(x, \mathbf{Y}) = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ f(x, y_1, y_2, \dots, y_n) \end{bmatrix}, \qquad \mathbf{Y}(x^{(0)}) = \begin{bmatrix} y_1^{(0)} \\ y_2^{(0)} \\ \vdots \\ y_n^{(0)} \end{bmatrix}$$

Example 3 - a harmonic oscilator (dumped oscillations)

Consider the equation $y'' + 2y' + y = e^{-t}$ with initial cond. $y(0) = 2,\ y'(0) = -4$.
Find the approximate solution at time $t = 0.2$. Use Euler method with $h = 0.1$.

The second-order problem has to be formulated as two first-order equations: set $y_1 = y$ and $y_2 = y'$ (i.e. use 2 scalar functions: y_1 represents an amplitude and y_2 a velocity). We have $y_1' = y_2$ and $y_2' = e^t - 2y_2 - y_1$:

$$\mathbf{Y}' = \begin{bmatrix} y_2 \\ e^{-t} - 2y_2 - y_1 \end{bmatrix}, \qquad \mathbf{Y}(0) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$h = 0.1,\ \ t^{(0)} = 0,\ \ \mathbf{Y}^{(0)} = (2, -4)^T$,

$$\mathbf{K} = \mathbf{F}(x^{(0)}, \mathbf{Y}^{(0)}) = \begin{bmatrix} -4 \\ e^0 - 2 \cdot (-4) - 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$$

$t^{(1)} = t^{(0)} + h = 0.1$

$$\mathbf{Y}^{(1)} = \mathbf{Y}^{(0)} + h\,\mathbf{K} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} + 0.1 \begin{bmatrix} -4 \\ 7 \end{bmatrix} = \begin{bmatrix} 1.6 \\ -3.3 \end{bmatrix}$$

$$\mathbf{K} = \mathbf{F}(x^{(1)}, \mathbf{Y}^{(1)}) = \begin{bmatrix} -3.3 \\ e^{-0.1} - 2 \cdot (-3.3) - 1.6 \end{bmatrix} = \begin{bmatrix} -3.3000 \\ 5.9048 \end{bmatrix}$$

$t^{(2)} = t^{(1)} + h = 0.2$

$$\mathbf{Y}^{(2)} = \mathbf{Y}^{(1)} + h\,\mathbf{K} = \begin{bmatrix} 1.6 \\ -3.3 \end{bmatrix} + 0.1 \begin{bmatrix} -3.3000 \\ 5.9048 \end{bmatrix} = \begin{bmatrix} 1.2700 \\ -2.7095 \end{bmatrix}$$

At time $t = 0.2$, the amplitude $y(0.2)$ is approximately 1.2700 and the velocity $y'(0.2)$ is approximately -2.7095. (The exact solution is $y(t) = (2 - 2t + 0.5t^2)\, e^{-t}$ and $y(0.2) = 1.3263$.)

Example 4

Consider Cauchy problem

$(x-1)y''' + 2xy'' + 5 \;=\; 2x^2 y'' + (x-1)\sqrt{(y')^2 - 2}$

with initial conditions $y(0) = 0,\; y'(0) = 2,\; y''(0) = -1$.

a) Find a domain where existence of a unique solution of the problem is guaranteed.

b) Compute an approximate value of $y'(0.1)$ using Euler method.

The solution

First of all, express the equation in normal (canonical) form:

$$y''' \;=\; \sqrt{(y')^2 - 2} \;+\; 2x\,y'' \;-\; \frac{5}{x-1}$$

Now set $y_1 = y,\; y_2 = y',\; y_3 = y''$ and transform it to the first-order system:

$$\mathbf{Y}' = \begin{bmatrix} y_2 \\ y_3 \\ \sqrt{(y_2)^2 - 2} \;+\; 2x\,y_3 \;-\; \dfrac{5}{x-1} \end{bmatrix}, \qquad \mathbf{Y}(0) = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

a) Functions $y_2,\; y_3$ and $\sqrt{(y_2)^2 - 1} + 2x\,y_3 - \frac{5}{x-1}$ and their derivatives with respect to y_i $\left(\frac{\partial f_3}{\partial y_2} = \frac{y_2}{\sqrt{(y_2)^2-2}}\right)$ are continuous for $x \neq 1$ a $y_2 \notin \langle -\sqrt{2}, \sqrt{2} \rangle$, i.e on the domains

$$\Omega_1 = (-\infty, 1) \times R \times (-\infty, -\sqrt{2}) \times R \quad , \quad \Omega_2 = (-\infty, 1) \times R \times (\sqrt{2}, \infty) \times R$$

$$\Omega_3 = (1, \infty) \times R \times (-\infty, -\sqrt{2}) \times R \quad , \quad \Omega_4 = (1, \infty) \times R \times (\sqrt{2}, \infty) \times R$$

The initial condition $[0, 0, 2, -1]$ is situated in the domain Ω_2, and so the domain, where existence of a unique solution is guaranteed, is Ω_2 .

b) We have $x^{(0)} = 0,\; \mathbf{Y}^{(0)} = (0, 2, -1)^T$ and we choose $h = 0.1$:

$$\mathbf{K} = \mathbf{F}(x^{(0)}, \mathbf{Y}^{(0)}) = \begin{bmatrix} 2 \\ -1 \\ \sqrt{2^2 - 2} \;+\; 2 \cdot 0 \cdot (-1) \;-\; \dfrac{5}{0-1} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ \sqrt{2} \;+\; 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6.4142 \end{bmatrix}$$

$$\mathbf{Y}^{(1)} = \mathbf{Y}^{(0)} + h\,\mathbf{K} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + 0.1 \begin{bmatrix} 2 \\ -1 \\ 6.4142 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 1.9 \\ -0.3586 \end{bmatrix}$$

The value of $y'(0.1)$ is approximately $y_2^{(1)} = 1.9$.