ODE - boundary value problems

The theory (very short excerpts from lectures)

Second-order boundary value problem

We want to find a solution $y(x)$ of a second-order, self-adjoint boundary value problem with Dirichlet boundary condition on the interval $\langle a, b\rangle$:

$$-(p(x)\, y'(x))' \;+\; q(x)\, y(x) \;=\; f(x)\;, \quad y(a) = y_0,\; y(b) = y_n \tag{1}$$

Existence and uniqueness of the (exact) solution

Sufficient conditions for existence of a unique solution of the problem (1):

- functions $p(x)$, $p'(x)$, $q(x)$, $f(x)$ are continuous on the interval $\langle a, b\rangle$, and
- $p(x) > 0$, $q(x) \geq 0$ on $\langle a, b\rangle$

Numerical solution by the finite-difference method

Choose a suitable step size h, put $n = \frac{b-a}{h}$ and define $n-1$ equidistant nodes inside the interval $\langle a, b\rangle$: $a = x_0 < x_1 < \dots x_n = b$, $x_{i+1} - x_i = h$ for all $i = 0, 1, 2, \dots n-1$.
Let us denote by y_i an approximate value of $y(x_i)$. Values y_0 and y_n are given as the boundary conditions, the rest of the values y_i for $i = 1, 2, \dots n-1$ can be computed from a system of linear equations

$$-p_{i-\frac{1}{2}}\; y_{i-1} + (p_{i-\frac{1}{2}} + h^2 q_i + p_{i+\frac{1}{2}})\; y_i - p_{i+\frac{1}{2}}\; y_{i+1} = h^2 f_i \tag{2}$$

where the following notation was used

$q_i = q(x_i)\;, \quad f_i = f(x_i)\;, \quad p_{i\pm\frac{1}{2}} = p(x_i \pm \frac{h}{2})$

If $p(x) > 0$ and $q(x) > 0$ on $\langle a, b\rangle$, then the matrix of the system is strictly diagonal dominant – so it is nonsingular and the system can be solved by Jacobi or Gauss-Seidel iterative method.

Inference of the system (2) from the exact equation (1)

The equation (1) is expressed in every inner node x_i for $i = 1, 2, \dots n-1$:

$$-(p(x_i)\, y'(x_i))' \;+\; q(x_i)\, y(x_i) \;=\; f(x_i)\;. \tag{3}$$

Then the first derivatives are approximated by central differences with $\frac{h}{2}$, in two steps:

1. Denote $z(x) = p(x)\, y'(x)$ and use the central difference: $z'(x_i) \approx \frac{z(x_i+h/2) - z(x_i-h/2)}{h}$, then the first term of equation (3) can be approximated as $-(p(x_i)\, y'(x_i))' = -z'(x_i) \approx$ $\approx -\frac{1}{h}\,[\,p(x_i + \frac{h}{2})\, y'(x_i + \frac{h}{2}) - p(x_i - \frac{h}{2})\, y'(x_i - \frac{h}{2})\,] = \frac{1}{h}\,[\,p_{i-\frac{1}{2}}\, y'(x_i - \frac{h}{2}) - p_{i+\frac{1}{2}}\, y'(x_i + \frac{h}{2})\,]$.
2. Use central differences at mid-points: $y'(x_i + \frac{h}{2}) \approx \frac{y(x_i+h) - y(x_i)}{h}$, $\; y'(x_i - \frac{h}{2}) \approx \frac{y(x_i) - y(x_i-h)}{h}$, then

$$-(p(x_i)\, y'(x_i))' \approx \frac{1}{h^2}\,[\,-p_{i+\frac{1}{2}}\; y(x_i+h) + (p_{i-\frac{1}{2}} + p_{i+\frac{1}{2}})\; y(x_i) - p_{i-\frac{1}{2}}\; y(x_i-h)\,]\;. \tag{4}$$

System (2) is obtained by substitution of (4) to (3), using $y_i \approx y(x_i)$ and rearranging.

Problem 1 - a harmonic oscilator (dumped oscillations)
Consider the equation $y'' + 2y' + y = e^{-t}$ with boundary conditions
$y(0) = 2,\ y(2) = 0$. (Exact solution is $y(t) = (2 - 2t + 0.5t^2)\, e^{-t}$.)
Find numerically an approximate value of the solution $y(0.2)$ at the time $t = 0.2$.

The solution

First of all, the given equation need to be transformed to the self-adjoint form (1).
This can be done by the following four steps:

- perform the differentiation of the first term of (1):

$$-p(x)\, y''(x) \;-\; p'(x)\, y'(x) \;+\; q(x)\, y(x) \;=\; f(x) \tag{5}$$

- multiply the given equation by $-p(x)$ (variable t is now renamed to x):

$$-p(x)\, y''(x) \;-\; 2\, p(x)\, y'(x) \;-\; p(x)\, y(x) \;=\; -p(x)\, e^{-x} \tag{6}$$

- compare the coefficients of (5) and (6):

  $p'(x) = 2\, p(x),\ \ q(x) = -p(x),\ \ f(x) = -p(x)\, e^{-x}$

- solve for p, q and f (hint: suppose $p(x) = e^{cx}$, then $p'(x) = c\, e^{cx} = c\, p(x)$):

  $p(x) = e^{2x},\ \ q(x) = -e^{2x},\ \ f(x) = -e^{2x}\, e^{-x} = -e^{x}$

The self-adjoint form is $-(e^{2t} y'(t))' - e^{2t} y(t) = -e^{t}$.

This problem does not comply with the conditions sufficient for the existence of unique solution listed above, so it is not guaranteed that we obtain a meaningful result when solving the equation numerically. Nevertheless, we try our hand at the numerical solution of this illustrative problem.

In order to find approximate solution at $t = 0.2$, we have to compute approximate solution on the whole interval. First of all let us divide the interval with step size $h = 0.2$ (it has to be chosen so that both endpoints of the interval are nodes of the mesh) and prepare coefficients needed for assembling the system of equations into Table 1.

Process of computation of the coefficients in Table 1:
We have $p(t) = e^{2t}$, $q(t) = -e^{2t}$ a $f(t) = -e^{t}$,

$p_{\frac{1}{2}} = p(0.1) = e^{0.2} = 1.2214$
$p_{1\frac{1}{2}} = p(0.3) = e^{0.6} = 1.8221$
$\dots$

$h^2 q_1 = 0.2^2 \cdot q(0.2) = 0.04 \cdot (-e^{0.4}) = -0.0597$
$h^2 q_2 = 0.2^2 \cdot q(0.4) = 0.04 \cdot (-e^{0.8}) = -0.0890$
$\dots$

$h^2 f_1 = 0.2^2 \cdot f(0.2) = 0.04 \cdot (-e^{0.2}) = -0.0489$
$h^2 f_2 = 0.2^2 \cdot f(0.4) = 0.04 \cdot (-e^{0.4}) = -0.0597$
$\dots$

Now we use the prepared coefficients for assembling 9 equations for 9 unknowns y_1 to y_9 :

i	t_i	$p_{i\pm\frac{1}{2}}$	$h^2 q_i$	$h^2 f_i$
	0.1	1.2214		
1	0.2		-0.0597	-0.0489
	0.3	1.8221		
2	0.4		-0.0890	-0.0597
	0.5	2.7183		
3	0.6		-0.1328	-0.0729
	0.7	4.0552		
4	0.8		-0.1981	-0.0890
	0.9	6.0496		
5	1.0		-0.2956	-0.1087
	1.1	9.0250		
6	1.2		-0.4409	-0.1328
	1.3	13.4637		
7	1.4		-0.6578	-0.1622
	1.5	20.0855		
8	1.6		-0.9813	-0.1981
	1.7	29.9641		
9	1.8		-1.4639	-0.2420
	1.9	44.7012		

Table 1: **Problem 1.** Coefficients needed for assembling the system of equations. Coefficients for the second equation are printed in color.

the first equation (for $i = 1$):

$-1.2214\ y_0 + (1.2214 - 0.0597 + 1.8221)\ y_1 - 1.8221\ y_2 = -0.0489$

substitute the boundary value $y_0 = 2$ and move the corresponding term

to the right hand side:

$2.9838\ y_1 - 1.8221\ y_2 = -0.0489 + 2 \cdot 1.2214 = 2.3939$

the second equation (for $i = 2$):

$-1.8221\ y_1 + (1.8221 - 0.0890 + 2.7183)\ y_2 - 2.7183\ y_3 = -0.0597$

$-1.8221\ y_1 + 4.4514\ y_2 - 2.7183\ y_3 = -0.0597$

. . . etc.

the last equation (for $i = 9$):

$-29.9641\ y_8 + (29.9641 - 1.4639 + 44.7012)\ y_9 - 44.7012\ y_{10} = -0.2420$

substitute the boundary value $y_{10} = 0$:

$-29.9641\ y_8 + 73.2014\ y_9 = -0.2420$

The solution of the system of equations is a vector
$Y = (1.3259, 0.8574, 0.5373, 0.3230, 0.1836, 0.0961, 0.0442, 0.0161, 0.0033)^T$.
An approximate value of $y(0.2)$ is $y_1 = 1.3259$
(for comparison: exact value of $y(0.2)$ is 1.3263).

Problem 2

Consider the equation

$$-(x^2y'(x))' + \frac{x^3}{2+x}y(x) = 4 + x$$

with boundary conditions $y(-5) = -2,\ y(-3) = 2.$

a) Prove the existence of a unique solution of the given problem.

b) Choose step size of $h = 0.4$ and using the finite difference method put together the system of linear equations for computing an approximate solution at the nodes of the mesh.

The solution

Using the notation of the general problem (1) we have

$p(x) = x^2$

$q(x) = \dfrac{x^3}{2+x}$

$f(x) = 4 + x$

a) Verify the conditions sufficient for the existence of an unique solution of the problem:

- functions x^2, $2x$, $\dfrac{x^3}{2+x}$, $4 + x$ are continuous in the interval $\langle -5, -3\rangle$,
- $x^2 > 0$, $\dfrac{x^3}{2+x} \geq 0$ in $\langle -5, -3\rangle$,

so there exists a unique solution of the given problem.

b) First divide the interval with the step size $h = 0.4$ and prepare coefficients needed for assembling the system of equations into table:

i	x_i	$p_{i\pm\frac{1}{2}}$	h^2q_i	h^2f_i
	-4.8	23.04		
1	-4.6		5.9899	-0.096
	-4.4	19.36		
2	-4.2		5.3882	-0.032
	-4.0	16.00		
3	-3.8		4.8775	0.032
	-3.6	12.96		
4	-3.4		4.4919	0.096
	-3.2	10.24		

Table 2: **Problem 2**. Coefficients needed for assembling the system of equations.

Computation of the coefficients:

$p_{\frac{1}{2}} = p(-4.8) = (-4.8)^2 = 23.04$

$p_{1\frac{1}{2}} = p(-4.4) = (-4.4)^2 = 19.36$

$\dots$

$h^2q_1 = 0.4^2 \cdot q(-4.6) = 0.16 \cdot \frac{(-4.6)^3}{2-4.6} = 5.9899$

$h^2q_2 = 0.4^2 \cdot q(-4.2) = 0.16 \cdot \frac{(-4.2)^3}{2-4.2} = 5.3882$

$\dots$

$h^2 f_1 = 0.4^2 \cdot f(-4.6) = 0.16 \cdot (4 - 4.6) = -0.096$
$h^2 f_2 = 0.4^2 \cdot f(-4.2) = 0.16 \cdot (4 - 4.2) = -0.032$
$\ldots$

Use the coefficitents for assembling 4 equations for 4 unknowns y_1 až y_4 :

the first equation ($i = 1$):

$-23.04\ y_0 + (23.04 + 5.9899 + 19.36)\ y_1 - 19.36\ y_2 = -0.096$

substitute the boundary value $y_0 = -2$ and move to the right hand side:

$48.3899\ y_1 - 19.36\ y_2 = -0.096 + 23.04 \cdot (-2) = -46.176$

the second equation ($i = 2$):

$-19.36\ y_1 + (19.36 + 5.3882 + 16.00)\ y_2 - 16.00\ y_3 = -0.032$

$-19.36\ y_1 + 40.7482\ y_2 - 16.00\ y_3 = -0.032$

the third equation ($i = 3$):

$-16.00\ y_2 + (16.00 + 4.8775 + 12.96)\ y_3 - 12.96\ y_4 = 0.032$

$-16.00\ y_2 + 33.8375\ y_3 - 12.96\ y_4 = 0.032$

the fourth equation ($i = 4$):

$-12.96\ y_3 + (12.96 + 4.4919 + 10.24)\ y_4 - 10.24\ y_5 = 0.096$

substitute the boundary value $y_5 = 2$ and move to the right hand side:

$-12.96\ y_3 + 27.6919\ y_4 = 0.096 + 10.24 \cdot 2 = 20.576$

Resulting system of equations in matrix notation:

$$\begin{bmatrix} 48.3899 & -19.36 & 0 & 0 \\ -19.36 & 40.7482 & -16.00 & 0 \\ 0 & -16.00 & 33.8375 & -12.96 \\ 0 & 0 & -12.96 & 27.6919 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -46.176 \\ -0.032 \\ 0.032 \\ 20.576 \end{bmatrix}$$

The solution of the system of equations is a vector

$Y =$ (-1.1719, -0.5440, 0.0345, 0.7592)^T

representing approximate values of the solution at inner nodes of the mesh, it means approximate values of $y(-4.6)$, $y(-4.2)$, $y(-3.8)$ and $y(-3.4)$.