ODE – initial value (or Cauchy) problems – Theory

(excerpts from lectures)

First-order initial value problem – recapitulation

$$\mathbf{Y}'(x) = \mathbf{F}(x, \mathbf{Y}(x)) \quad \text{with an initial condition } \mathbf{Y}(x_0) = \mathbf{Y}^{(0)} \,, \tag{1}$$

where

$$\mathbf{Y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}, \ \mathbf{Y}'(x) = \begin{bmatrix} y_1'(x) \\ y_2'(x) \\ \vdots \\ y_n'(x) \end{bmatrix}, \ \mathbf{F}(x, \mathbf{Y}) = \begin{bmatrix} f_1(x, y_1, y_2, \dots, y_n) \\ f_2(x, y_1, y_2, \dots, y_n) \\ \vdots \\ f_n(x, y_1, y_2, \dots, y_n) \end{bmatrix}$$

The system of (1) is called *linear*, if functions f_i are linear with respect to all variables y_j , i.e., they have the form

$$f_i(x, y_1, y_2, \dots, y_n) = g_{i0}(x) \, + \, g_{i1}(x) \, y_1 \, + \, g_{i2}(x) \, y_2 \, + \dots + \, g_{in}(x) \, y_n \ , \quad i = 1, \dots n \ . \tag{2}$$

Existence and uniqueness of the (exact) solution

The standard theorem, which formulates *sufficient* conditions for existence and uniqueness of the solution of the initial value problem (1):

Theorem 1

Suppose that functions f_i are continuous in some region D defined by $a \le x \le b$, $a_j < y_j < b_j$ for $j = 1, \dots n$ and that F satisfies *Lipschitz condition* with respect to $\mathbf{Y}$ in D, i. e., there exists a constant L such that

$$\|F(x, \mathbf{Y}) - F(x, \mathbf{Z})\| \ \le \ L \, \|\mathbf{Y} - \mathbf{Z}\| \quad \forall \ (x, \mathbf{Y}), \, (x, \mathbf{Z}) \in D \ . \tag{3}$$

Then for any $(x^{(0)}, \mathbf{Y}^{(0)}) \in D$ there exists a unique maximal solution $\mathbf{Y}(x) \subset D$ of the problem (1). Moreover, if $D = I \times R^n$, then the unique maximal solution is defined on the whole interval I.

For a **linear system** (2) the following holds:

Suppose that all functions $g_{ij}(x)$ are continuous on an interval $I =< a, b >$. Then the assumptions of Theorem 1 are satisfied in $D = I \times R^n$ with $L = \max_{x \in I} |g_{ij}(x)|$, so the unique maximal solution is defined on the whole interval I.

The Lipschitz condition (3) can be interpreted as requiring a little more than continuity but a little less than differentiability:

Theorem 2

Suppose that functions f_i are continuous in some region D defined by $a \le x \le b$, $a_j < y_j < b_j$ for $j = 1, \dots n$, and that they have also continuous partial derivatives with respect to y_j, $j = 1, \dots, n$ there. Then the inequality (3) is satisfied in D with $L = \sup_D \| \dfrac{\partial f_i}{\partial y_j} \|$ and if $L < \infty$, then the assumptions of Theorem 1 hold in D.

Throughout the text, we assume that the problem (1) satisfies the assumptions of Theorem 1 in some region D and $(x^{(0)}, \mathbf{Y}^{(0)}) \in D$.

It also implicates that the problem is *well-posed*, which means (roughly speaking) that it's solution depends continuously on the given data: that a small perturbation of $\mathbf{F}$ or $\mathbf{Y}^{(0)}$ leads to a small change of the solution.

General one-step method

In this text we consider *one-step methods* only, which have a form of

$$\mathbf{Y}^{(k+1)} = \mathbf{Y}^{(k)} + h\ \Phi(\,\mathbf{Y}^{(k)}, \mathbf{Y}^{(k+1)},\, x_k\,, h)\ , \tag{4}$$

where h is the step-size, $x_k = x_0 + k\,h$ and $\mathbf{Y}^{(k)}$ is the numerical approximation of the exact solution $\mathbf{Y}(x_k)$ of the problem (1).

Examples of one-step methods:

- Explicit Euler method: $\Phi(\,\mathbf{Y}^{(k)}, \mathbf{Y}^{(k+1)},\, x_k\,, h) = \mathbf{F}(\,x_k\,, \mathbf{Y}^{(k)})$
- Implicit Euler method: $\Phi(\,\mathbf{Y}^{(k)}, \mathbf{Y}^{(k+1)},\, x_k\,, h) = \mathbf{F}(\,x_k + h\,, \mathbf{Y}^{(k+1)})$
- Collatz (or midpoint) method: $\Phi(\,\mathbf{Y}^{(k)}, \mathbf{Y}^{(k+1)},\, x_k\,, h) = \mathbf{F}(\,x_k + \frac{h}{2}\,, \mathbf{Y}^{(k)} + \frac{h}{2}\,\mathbf{F}(\,x_k\,, \mathbf{Y}^{(k)}))$

Convergence, consistency, stability

$$\begin{tikzcd}
\boxed{\text{differential equation}} \arrow[rr, leftrightarrow, "\textbf{consistency}"] \arrow[d, "\textbf{exist. and uniq. of solution}"] & & \boxed{\text{discretized equation}} \arrow[d, "\textbf{stability}"] \\
\boxed{\text{exact solution}} \arrow[rr, leftrightarrow, "\textbf{convergence}"] & & \boxed{\text{numerical solution}}
\end{tikzcd}$$

Convergence

Global truncation error (GTE) of the approximate solution $\mathbf{Y}^{(k)}$ at $x^{(k)}$ is defined as

$$e_k = \|\,\mathbf{Y}^{(k)}\ -\ \mathbf{Y}(x^{(k)})\|\,.$$

The numerical solution should approach the exact one (converge to it) as the mesh-size tends to zero, it is if for some norm of the vector $\mathbf{e} = (e_1, \dots e_n)^T$ of global errors

$$\|\mathbf{e}\| \to 0 \quad \text{as} \quad h \to 0\ .$$

The method is of p-th order of accuracy, if some norm of a vector $\mathbf{e}$ is $\mathcal{O}(h^p)$.

Consistency of discretized equation with differential equation

The discretization of the differential equation should become exact, as the mesh-size tends to zero. Consistency error (for one-step method):

$$\eta_k = \left\| \frac{\mathbf{Y}(x_{k+1}) - \mathbf{Y}(x_k)}{h} - \ \Phi(\,\mathbf{Y}(x_k), \mathbf{Y}(x_{k+1}),\, x_k\,, h) \right\| \tag{5}$$

– it is the error in discretized equation, if exact solution evaluated at mesh-points is substituted into it. It measures the extent to which the true solution satisfies the discrete equation. Consistency errors should vanish, as the mesh-size tends to zero.

Stability

Numerical errors that are generated as a consequence of using discretized equation, should be held under control. There are several different definitions of stability which put this idea into more specific terms.

Lax equivalence theorem

For *linear* well-posed initial value problem and *consistent* finite difference approximation of it, *stability* is necessary and sufficient condition for *convergence*.

Note: for nonlinear problems this equivalence does not hold. However, we probably cannot expect good behaviour of any method which is not convergent for linear problems, so consistency and stability of methods are important even if nonlinear problems are solved, when these two properties cannot guarantee convergency.

Examples

The theory will be illustrated on a scalar equation as the simplest case of the system (1):

$$y'(x) = f(x, y(x)) \;, \qquad y(x_0) = y^{(0)} \;. \tag{6}$$

1. Analysis of explicit Euler method $x^{(k+1)} = x^{(k)} + h, \quad y^{(k+1)} = y^{(k)} + \; h \; f(x^{(k)}, y^{(k)})$

Convergence – recapitulation:

Local error is $\mathcal{O}(h^2)$,
Global error is $\mathcal{O}(h)$.

Consistency

Consistency error (5) is

$$\eta_k = \left\| \frac{y(x_{k+1}) - y(x_k)}{h} - \; f(x_k, y(x_k)) \right\|$$

Taylor expansion gives
$y(x_{k+1}) \equiv y(x_k + h) = y(x_k) + h\, y'(x_k) + \mathcal{O}(h^2) = y(x_k) + h\, f(x_k, y(x_k)) + \mathcal{O}(h^2)$
so $y(x_{k+1}) - y(x_k) = h\, f(x_k, y(x_k)) + \mathcal{O}(h^2)$ and after substitution we have

$$\eta_k = \left\| \frac{h\, f(x_k, y(x_k)) + \mathcal{O}(h^2)}{h} - f(x_k, y(x_k)) \right\| = \|\, f(x_k, y(x_k)) + \mathcal{O}(h) - f(x_k, y(x_k)) \,\| = \mathcal{O}(h) \,,$$

which converges to zero as $h \to 0$.

Stability

Stability will be studied only on a standard model equation

$$y'(x) = -a\, y(x) \,, \quad y(x_0) = y^{(0)} \,, \quad a > 0 \tag{7}$$

However, if a method is not stable on this simple linear equation, we probably cannot expect its good behaviour on other equations, too.

The exact solution of equation (7) is $\mathrm{e}^{-a(x-x_0)}$, which tends to zero as $x \to \infty$.

Euler method:
$y^{(k+1)} = y^{(k)} + h(-a\,y^{(k)}) = (1 - h\,a)\,y^{(k)} = (1 - h\,a)^2\,y^{(k-1)} = \cdots = (1 - h\,a)^{k+1}\,y^{(0)}$

$y^{(k+1)}$ converges to zero if and only if the modulus of the *growth factor* $(1 - h\,a)$ is less than 1, which means $|ha| < 2$ (for $1 < |ha| < 2$ it alternates sign, however it is stable).

So explicit Euler method is only *conditionally stable*, see Figure 1.

conditional stability:

- existence of a *critical time step* beyond which numerical instabilities occur,
- is typical for explicit methods

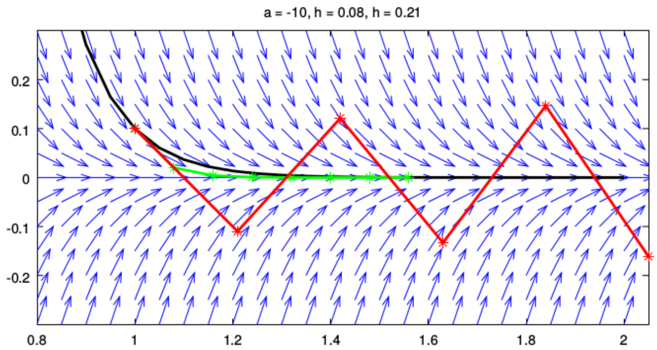

Figure 1: Problem $y' = -10\,y\,,\ y(1) = 0.1$ solved by explicit Euler method. Black line: exact solution, green line: numerical solution with $h = 0.08$, red line: numerical solution with $h = 0.21$. Blue arrows at background represent directional field for the given equation.

2. Stability of implicit Euler method studied on a standard model equation (7) only
Implicit Euler method: $y^{(k+1)} = y^{(k)} + h(-a\,y^{(k+1)})$ and for $y^{(k+1)}$ the explicit formula is obtained

$$y^{(k+1)} = \frac{1}{1+ha}\,y^{(k)} = \left(\frac{1}{1+ha}\right)^2 y^{(k-1)} = \cdots = \left(\frac{1}{1+ha}\right)^{k+1} y^{(0)}$$

$y^{(k+1)}$ converges to zero for any choice of h, because the growth factor is allways less than 1 and implicit Euler method is *unconditionally stable.* This is typical behaviour of other implicit methods, too. Compare Figures 1 and 2 – the same problem is solved by explicit (Fig. 1) or implicit (Fig. 2) method.

3. Consistency of one-step method

Consistency error (5) is

$$\eta_k = \left\| \frac{y(x_{k+1}) - y(x_k)}{h} - \Phi(\,y(x_k), y(x_{k+1}),\, x_k\,, h) \right\| ,$$

the limit of the first term is $\lim\limits_{h\to 0} \frac{y(x_{k+1})-y(x_k)}{h} = y'(x_k) = f(x_k, y(x_k))$, so the consistency error tends to zero if and only if $\lim\limits_{h\to 0} \Phi(\,y(x_k), y(x_{k+1}),\, x_k\,, h) = f(x_k, y(x_k))$ for all x_k.

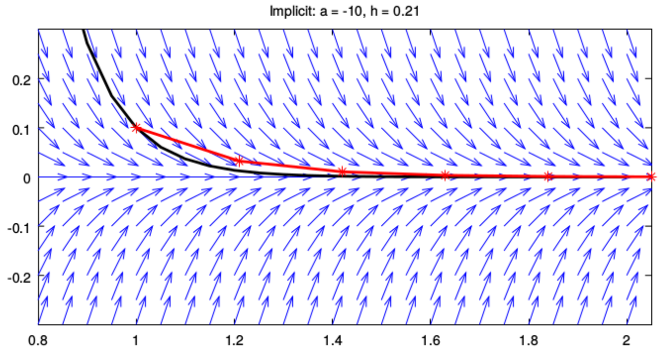

Figure 2: Problem $y' = -10\,y\,,\; y(1) = 0.1$ solved by implicit Euler method. Black line: exact solution, red line: numerical solution with $h = 0.21$. Blue arrows at background represent directional field for the given equation.

References

- J. D. Lambert: Numerical Methods for Ordinary Differential Systems, Willey & Sons, 1993
- Gilbert Strang: Computational Science and Engineering, online MIT courses
- D. N. Arnold: Stability, consistency, and convergence of numerical discretizations, `http://www-users.math.umn.edu/~arnold/papers/stability.pdf`