ODE - initial value (or Cauchy) problems

The theory (very short excerpts from lectures)

First-order initial value problem

We want to approximate the solution $\mathbf{Y}(x)$ of a system of first-order ordinary differential equations

$$\mathbf{Y}'(x) = \mathbf{F}(x, \mathbf{Y}(x)) \quad \text{with an initial condition } \mathbf{Y}(x^{(0)}) = \mathbf{Y}^{(0)}, \tag{1}$$

where

$$\mathbf{Y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}, \mathbf{Y}'(x) = \begin{bmatrix} y_1'(x) \\ y_2'(x) \\ \vdots \\ y_n'(x) \end{bmatrix}, \mathbf{F}(x, \mathbf{Y}) = \begin{bmatrix} f_1(x, y_1, y_2, \ldots, y_n) \\ f_2(x, y_1, y_2, \ldots, y_n) \\ \vdots \\ f_n(x, y_1, y_2, \ldots, y_n) \end{bmatrix}$$

Existence and uniqueness of the (exact) solution

Let functions f_i are continuous in a domain Ω and have also continuous partial derivatives $\dfrac{\partial f_i}{\partial y_j}$, $i, j = 1, \ldots, n$ there. Then every point $[x^0, \mathbf{Y}^0] \in \Omega$ determines a unique maximal solution, such that $\mathbf{Y}(x^0) = \mathbf{Y}^0$ and $[x, \mathbf{Y}(x)] \subset \Omega$.

The system (1) is called *linear*, if functions f_i are linear according to all variables y_j , i.e. they have the form $f_i(x, y_1, y_2, \ldots, y_n) = g_{i0}(x) + g_{i1}(x)\, y_1 + g_{i2}(x)\, y_2 + ... + g_{in}(x)\, y_n$.

For a linear system the following holds:

Let all functions $g_{ij}(x)$ are continuous on an interval I.
Then, for every $x^0 \in I$, the point $[x^0, \mathbf{Y}^0]$ determines a unique maximal solution defined on the whole interval I, such that $\mathbf{Y}(x^0) = \mathbf{Y}^0$.

Explicit Euler's method

choose a step size h and for $i = 0, 1, 2, \ldots$

1. compute the derivative $\mathbf{K}$ of the vector function $\mathbf{Y}$ as
   $\mathbf{K} = \mathbf{F}(x^{(i)}, \mathbf{Y}^{(i)})$
2. put
   $x^{(i+1)} = x^{(i)} + h$
   $\mathbf{Y}^{(i+1)} = \mathbf{Y}^{(i)} + h\,\mathbf{K}$

Implicit Euler's method

choose a step size h and for $i = 0, 1, 2, \ldots$

1. put $x^{(i+1)} = x^{(i)} + h$
2. compute $\mathbf{Y}^{(i+1)}$ from the equation
   $\mathbf{Y}^{(i+1)} = \mathbf{Y}^{(i)} + h\,\mathbf{F}(x^{(i+1)}, \mathbf{Y}^{(i+1)})$
   (using fixed point iterations or Newton method, for instance)

Collatz (or midpoint) method

choose a step size h and for $i = 0, 1, 2, \dots$

1. compute an auxiliary point $[x_p, \mathbf{Y}_p]$ using Euler method with half-step:

   $\mathbf{K}_1 = \mathbf{F}(x^{(i)},\, \mathbf{Y}^{(i)})$

   $x_p = x^{(i)} + \frac{1}{2}h$

   $\mathbf{Y}_p = \mathbf{Y}^{(i)} + \frac{1}{2}h\,\mathbf{K}_1$

2. compute the derivative $\mathbf{K}_2$ at the auxiliary point $[x_p, \mathbf{Y}_p]$ as

   $\mathbf{K}_2 = \mathbf{F}(x_p,\, \mathbf{Y}_p)$

3. put

   $x^{(i+1)} = x^{(i)} + h$

   $\mathbf{Y}^{(i+1)} = \mathbf{Y}^{(i)} + h\,\mathbf{K}_2$

Higher-order initial value problems

We want to approximate the solution $y(x)$ of the differential equation of the n-th order

$$y^n(x) = f(x, y, y', y'', \dots y^{n-1}) \quad \text{with initial conditions}$$
$$y(x^{(0)}) = y_1^{(0)}, \quad y'(x^{(0)}) = y_2^{(0)}, \quad \dots \quad y^{n-1}(x^{(0)}) = y_n^{(0)} \tag{2}$$

In order to be able to use Euler or Collatz method, we need to represent this diferential equation of n-th order as n first-order diferential equations. Introducing auxiliary variables $y_1 = y$, $y_2 = y'$, $y_3 = y''$, $\dots y_n = y^{n-1}$ into equation (2) leads to a system

$$\mathbf{F}(x, \mathbf{Y}) = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ f(x, y_1, y_2, \dots, y_n) \end{bmatrix}, \qquad \mathbf{Y}(x^{(0)}) = \begin{bmatrix} y_1^{(0)} \\ y_2^{(0)} \\ \vdots \\ y_n^{(0)} \end{bmatrix}$$

Example 1

Consider Cauchy problem $\quad y' = \dfrac{y}{x^2}\,, \quad y(1) = 2\,.$

1) Find a domain where existence of a unique solution of the problem is guaranteed.

2) Compute an approximate value of $y(1.4)$ using:
   a) Explicit Euler method with step size $h = 0.2$,
   b) Implicit Euler method with step size $h = 0.2$,
   c) Explicit and Implicit Euler method with step size $h = 0.1$,
   d) Collatz method with step size $h = 0.2$.

The solution

1) Functions $f(x,y) = \dfrac{y}{x^2}$, $\dfrac{\partial f}{\partial y} = \dfrac{1}{x^2}$ are continuous everywhere with the exception of y-axis, so there are two domains, where an unique solution exists: $\Omega_1 = (-\infty, 0) \times (-\infty, \infty)$, $\Omega_2 = (0, \infty) \times (-\infty, \infty)$

As the initial condition $[1, 2]$ is situated in the domain Ω_2, the domain of existence and uniqueness of solution of the given problem is Ω_2. (However, the given equation is linear and so it would be sufficient to check the continuity of $f(x,y)$ according to x. In this linear case we also can specify the interval of maximal solution $I = (0, \infty)$).

2) The results are summarized in Table 1 and for explicit Euler method also depicted in Figure 1. These results show that Collatz method gives more precise solution than Euler method (both explicit and implicit), even in the case when for Collatz method, step size twice as long as for Euler was used (which represents comparable work).

Computation:

a) $h = 0.2, \quad x^{(0)} = 1, \quad y^{(0)} = 2$

$k = \dfrac{y^{(0)}}{(x^{(0)})^2} = \dfrac{2}{1^2} = 2$

$x^{(1)} = x^{(0)} + h = 1 + 0.2 = 1.2, \quad y^{(1)} = y^{(0)} + h\,k = 2 + 0.2 \cdot 2 = 2.4$

$k = \dfrac{y^{(1)}}{(x^{(1)})^2} = \dfrac{2.4}{(1.2)^2} = 1.6667$

$x^{(2)} = x^{(1)} + h = 1.2 + 0.2 = 1.4, \quad y^{(2)} = y^{(1)} + h\,k = 2.4 + 0.2 \cdot 1.6667 = 2.7333$

$y(1.4)$ is approximately equal to $y^{(2)} = 2.7333$.

b) There is no general explicit formula; in every iteration, we have to solve an equation $y^{(i+1)} = y^{(i)} + h\,f(x^{(i+1)}, y^{(i+1)})$; for this problem it is

$$y^{(i+1)} = y^{(i)} + h\,\frac{y^{(i+1)}}{(x^{(i+1)})^2} \ .$$

In the case of *linear* differential equation like this, however, we can express $y^{(i+1)}$ from the equation above explicitly:

$$y^{(i+1)} = \frac{(x^{(i+1)})^2}{(x^{(i+1)})^2 - h}\,y^{(i)} \ .$$

$h = 0.2, \quad x^{(0)} = 1, \quad y^{(0)} = 2$

$x^{(1)} = x^{(0)} + h = 1 + 0.2 = 1.2\ , \quad y^{(1)} = \frac{(x^{(1)})^2}{(x^{(1)})^2 - h}\,y^{(0)} = \frac{1.2^2}{1.2^2 - 0.2} \cdot 2 = 2.3226$

$x^{(2)} = x^{(1)} + h = 1.2 + 0.2 = 1.4\ , \quad y^{(2)} = \frac{(x^{(2)})^2}{(x^{(2)})^2 - h}\,y^{(1)} = \frac{1.4^2}{1.4^2 - 0.2} \cdot 2.3226 = 2.5865$

$y(1.4)$ is approximately equal to $y^{(2)} = 2.5865$.

c) Using similar process as in a), we obtain values presented at the second column of Table 1: $y(1.4)$ is approximately equal to $y^{(4)} = 2.6979$ for explicit Euler method and to $y^{(4)} = 2.6241$ for the implicit one.

d) $h = 0.2, \quad x^{(0)} = 1, \quad y^{(0)} = 2$

$$k_1 = \frac{y^{(0)}}{(x^{(0)})^2} = \frac{2}{1^2} = 2,$$
$$x_p = x^{(0)} + \tfrac{1}{2}h = 1 + 0.1 = 1.1, \qquad y_p = y^{(0)} + \tfrac{1}{2}h\,k_1 = 2 + 0.1 \cdot 2 = 2.2$$
$$k_2 = \frac{y_p}{x_p^2} = \frac{2.2}{1.1^2} = 1.8182$$
$$x^{(1)} = x^{(0)} + h = 1 + 0.2 = 1.2, \quad y^{(1)} = y^{(0)} + h\,k_2 = 2 + 0.2 \cdot 1.8182 = 2.3636$$

$$k_1 = \frac{y^{(1)}}{(x^{(1)})^2} = \frac{2.3636}{1.2^2} = 1.6414$$
$$x_p = x^{(1)} + \tfrac{1}{2}h = 1.2 + 0.1 = 1.3$$
$$y_p = y^{(1)} + \tfrac{1}{2}h\,k_1 = 2.3636 + 0.1 \cdot 1.6414 = 2.5278$$
$$k_2 = \frac{y_p}{x_p^2} = \frac{2.5278}{1.3^2} = 1.4957$$
$$x^{(2)} = x^{(1)} + h = 1.2 + 0.2 = 1.4$$
$$y^{(2)} = y^{(1)} + h\,k_2 = 2.3636 + 0.2 \cdot 1.4957 = 2.6628$$

$y(1.4)$ is approximately equal to $y^{(2)} = 2.6628$.

$x^{(i)}$	exact $y(x^{(i)})$	Euler Explic.	$h=0.1$ Implic.	Euler Explic.	$h=0.2$ Implic.	Collatz $h=0.2$
1	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000
1.1	2.1903	2.2000	2.1802			(2.2000)
1.2	2.3627	2.3818	2.3429	2.4000	2.3226	2.3636
1.3	2.5191	2.5472	2.4902			(2.5278)
1.4	2.6614	2.6979	2.6241	2.7333	2.5865	2.6628
1.5	2.7912	2.8356	2.7462			(2.7986)
1.6	2.9100	2.9616	2.8578	3.0122	2.8057	2.9115
1.7	3.0190	3.0773	2.9602			(3.0253)
1.8	3.1192	3.1838	3.0545	3.2476	2.9903	3.1209
1.9	3.2118	3.2821	3.1415			(3.2172)
2.0	3.2974	3.3730	3.2221	3.4480	3.1477	3.2992

Table 1: **Example 1**. The first column represents values of x, where the approximate solution is computed. At the second column there is exact solution, the third column presents approximate solution obtained by Euler method with step size $h = 0.1$, at the fourth column there is Euler method with the step size twice as big and the last column presents approximate solution obtained by Collatz method with step size $h = 0.2$. Results from the first four columns are depicted in Figure 1.

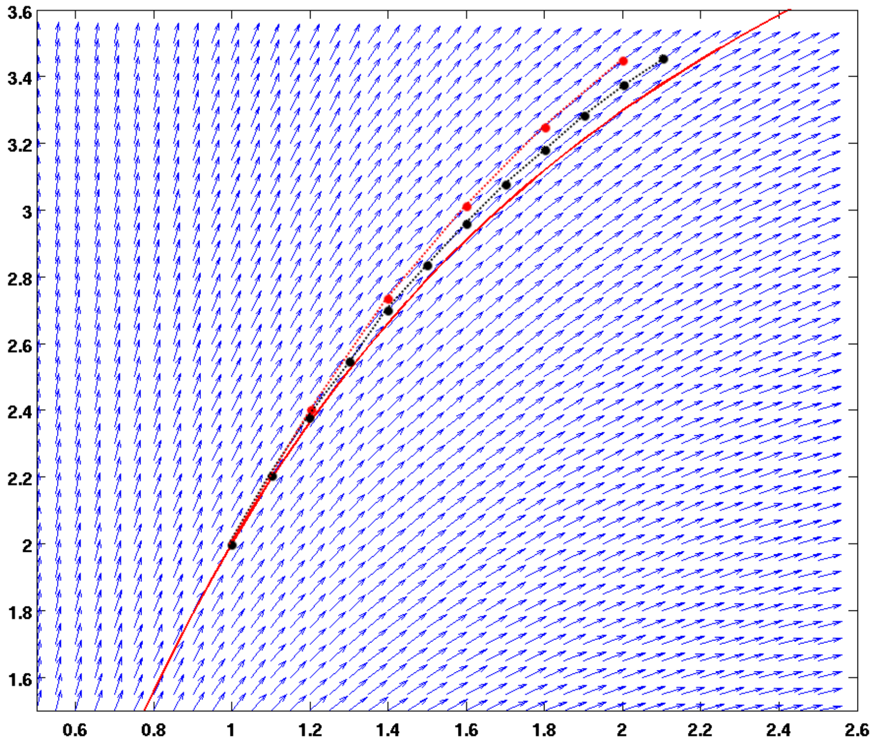

Figure 1: **Example 1**. Horizontal axis is x, vertical axis is y. Every blue arrow represent tangent vector to the integral curve passing through the matching point. The full red line represents the exact solution of the problem with a given initial condition $y(x) = 2\,e^{1-1/x}$ (it can be computed using a separation of variables). Red and black points represent approximation of the solution computed by Euler method with step size 0.2 and 0.1, respectively (see also Table 1).

Example 2

Consider Cauchy problem

$$\mathbf{Y}' = \begin{bmatrix} y_1 \sin(x) + y_3 \\ y_2 \ln(x+1) - 4 \\ 2y_1 \; - \; \dfrac{y_3}{x-2} \end{bmatrix}, \qquad \mathbf{Y}(1) = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

a) Check that the given problem has unique solution and find the interval I of its maximal solution.

b) Choose a step size $h = 0.1$ and compute an approximate value of $\mathbf{Y}(1.2)$ using Euler method.

c) Choose a step size $h = 0.2$ and compute an approximate value of $\mathbf{Y}(1.2)$ using Collatz method.

The solution

a) $x + 1 > 0 \;\Rightarrow\; x > -1, \quad x - 2 \neq 0 \;\Rightarrow\; x \neq 2 \qquad I_1 = (-1, 2),\; I_2 = (2, \infty)$
$x^{(0)} = 1 \in I_1 \Rightarrow$ interval of maximal solution is $(-1, 2)$.

b) $x^{(0)} = 1$, $\mathbf{Y}^{(0)} = (-1, 1, 2)^T$, $h = 0.1$:

$$\mathbf{K} = \mathbf{F}(x^{(0)}, \mathbf{Y}^{(0)}) = \begin{bmatrix} -1 \cdot \sin(1) + 2 \\ 1 \cdot \ln(1+1) - 4 \\ 2 \cdot (-1) \; - \; \dfrac{2}{1-2} \end{bmatrix} = \begin{bmatrix} -0.84147 + 2 \\ 0.69315 - 4 \\ -2 + 2 \end{bmatrix} = \begin{bmatrix} 1.1585 \\ -3.3068 \\ 0 \end{bmatrix}$$

$x^{(1)} = x^{(0)} + h = 1 + 0.1 = 1.1$

$$\mathbf{Y}^{(1)} = \mathbf{Y}^{(0)} + h\,\mathbf{K} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + 0.1 \begin{bmatrix} 1.1585 \\ -3.3068 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.8842 \\ 0.6693 \\ 2 \end{bmatrix}$$

$$\mathbf{K} = \mathbf{F}(x^{(1)}, \mathbf{Y}^{(1)}) = \begin{bmatrix} -0.8842 \cdot \sin(1.1) + 2 \\ 0.6693 \cdot \ln(1.1+1) - 4 \\ 2 \cdot (-0.8842) \; - \; \dfrac{2}{1.1-2} \end{bmatrix} = \begin{bmatrix} -0.7880 + 2 \\ 0.6693 \cdot 0.74194 - 4 \\ -1.7684 + 2.2222 \end{bmatrix} = \begin{bmatrix} 1.2120 \\ -3.5034 \\ 0.45380 \end{bmatrix}$$

$x^{(2)} = x^{(1)} + h = 1.1 + 0.1 = 1.2$

$$\mathbf{Y}^{(2)} = \mathbf{Y}^{(1)} + h\,\mathbf{K} = \begin{bmatrix} -0.8842 \\ 0.6693 \\ 2 \end{bmatrix} + 0.1 \begin{bmatrix} 1.2120 \\ -3.5034 \\ 0.4538 \end{bmatrix} = \begin{bmatrix} -0.7630 \\ 0.3190 \\ 2.0454 \end{bmatrix}$$

The value of $\mathbf{Y}(1.2)$ is approximately $\mathbf{Y}^{(2)} = (-0.7630, 0.3190, 2.0454)^T$.

c) $x^{(0)} = 1$, $\mathbf{Y}^{(0)} = (-1, 1, 2)^T$, $h = 0.2$:
$[x_p, \mathbf{Y}_p]$ is equal to $[x^{(1)}, \mathbf{Y}^{(1)}]$ from b), so we can use the value of derivative at this point, which we have already computed at b) above:

$$\mathbf{K}_2 = \mathbf{F}(x_p, \mathbf{Y}_p) = \begin{bmatrix} 1.2120 \\ -3.5034 \\ 0.45380 \end{bmatrix}$$

$$\mathbf{Y}^{(1)} = \mathbf{Y}^{(0)} + h\,\mathbf{K}_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + 0.2 \begin{bmatrix} 1.2120 \\ -3.5034 \\ 0.4538 \end{bmatrix} = \begin{bmatrix} -0.7576 \\ 0.2993 \\ 2.091 \end{bmatrix}$$

The value of $\mathbf{Y}(1.2)$ is approximately $\mathbf{Y}^{(1)} = (-0.7782,\ 0.2993,\ 2.091)^T$.

Example 3 - a harmonic oscilator (dumped oscillations)

Consider the equation $y'' + 2y' + y = e^{-t}$ with initial cond. $y(0) = 2,\ y'(0) = -4$.
Find the approximate solution at time $t = 0.2$. Use Euler method with $h = 0.1$.

The second-order problem has to be formulated as two first-order equations: put $y_1 = y$ and $y_2 = y'$ (i.e. use 2 scalar functions: y_1 represents an amplitude and y_2 a velocity). We have $y_1' = y_2$ and $y_2' = e^t - 2y_2 - y_1$:

$$\mathbf{Y}' = \begin{bmatrix} y_2 \\ e^{-t} - 2y_2 - y_1 \end{bmatrix}, \qquad \mathbf{Y}(0) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$h = 0.1,\ t^{(0)} = 0,\ \mathbf{Y}^{(0)} = (2, -4)^T$,

$$\mathbf{K} = \mathbf{F}(x^{(0)}, \mathbf{Y}^{(0)}) = \begin{bmatrix} -4 \\ e^0 - 2\cdot(-4) - 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$$

$t^{(1)} = t^{(0)} + h = 0.1$

$$\mathbf{Y}^{(1)} = \mathbf{Y}^{(0)} + h\,\mathbf{K} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} + 0.1 \begin{bmatrix} -4 \\ 7 \end{bmatrix} = \begin{bmatrix} 1.6 \\ -3.3 \end{bmatrix}$$

$$\mathbf{K} = \mathbf{F}(x^{(1)}, \mathbf{Y}^{(1)}) = \begin{bmatrix} -3.3 \\ e^{-0.1} - 2\cdot(-3.3) - 1.6 \end{bmatrix} = \begin{bmatrix} -3.3000 \\ 5.9048 \end{bmatrix}$$

$t^{(2)} = t^{(1)} + h = 0.2$

$$\mathbf{Y}^{(2)} = \mathbf{Y}^{(1)} + h\,\mathbf{K} = \begin{bmatrix} 1.6 \\ -3.3 \end{bmatrix} + 0.1 \begin{bmatrix} -3.3000 \\ 5.9048 \end{bmatrix} = \begin{bmatrix} 1.2700 \\ -2.7095 \end{bmatrix}$$

At time $t = 0.2$, the amplitude $y(0.2)$ is approximately 1.2700 and the velocity $y'(0.2)$ is approximately -2.7095. (The exact solution is $y(t) = (2 - 2t + 0.5t^2)\, e^{-t}$ and $y(0.2) = 1.3263$.)

Example 4

Consider Cauchy problem

$$(x-1)y''' + 2xy'' + 5 \;=\; 2x^2y'' + (x-1)\sqrt{(y')^2-2}$$

with initial conditions $y(0) = 0,\ y'(0) = 2,\ y''(0) = -1$.

a) Find a domain where existence of a unique solution of the problem is guaranteed.

b) Compute an approximate value of $y'(0.1)$ using Euler method.

The solution

First of all, express the equation in normal (canonical) form:

$$y''' \;=\; \sqrt{(y')^2-2} \;+\; 2x\,y'' \;-\; \frac{5}{x-1}$$

Now put $y_1 = y,\ y_2 = y',\ y_3 = y''$ and transform it to the first-order system:

$$\mathbf{Y}' = \begin{bmatrix} y_2 \\ y_3 \\ \sqrt{(y_2)^2-2} \;+\; 2x\,y_3 \;-\; \dfrac{5}{x-1} \end{bmatrix}, \qquad \mathbf{Y}(0) = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

a) Functions $y_2,\ y_3$ and $\sqrt{(y_2)^2-1} + 2x\,y_3 - \frac{5}{x-1}$ and their derivatives with respect to y_i ($\frac{\partial f_3}{\partial y_2} = \frac{y_2}{\sqrt{(y_2)^2-2}}$) are continuous for $x \neq 1$ a $y_2 \notin \langle -\sqrt{2}, \sqrt{2}\rangle$, i.e on the domains

$$\Omega_1 = (-\infty,1)\times R\times(-\infty,-\sqrt{2})\times R \quad , \quad \Omega_2 = (-\infty,1)\times R\times(\sqrt{2},\infty)\times R$$

$$\Omega_3 = (1,\infty)\times R\times(-\infty,-\sqrt{2})\times R \quad , \quad \Omega_4 = (1,\infty)\times R\times(\sqrt{2},\infty)\times R$$

The initial condition $[0,0,2,-1]$ is situated in the domain Ω_2, and so the domain, where existence of a unique solution is guaranteed, is Ω_2 .

b) We have $x^{(0)} = 0,\ \mathbf{Y}^{(0)} = (0,2,-1)^T$ and we choose $h = 0.1$:

$$\mathbf{K} = \mathbf{F}(x^{(0)},\mathbf{Y}^{(0)}) = \begin{bmatrix} 2 \\ -1 \\ \sqrt{2^2-2} \;+\; 2\cdot 0\cdot(-1) \;-\; \dfrac{5}{0-1} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ \sqrt{2} \;+\; 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6.4142 \end{bmatrix}$$

$$\mathbf{Y}^{(1)} = \mathbf{Y}^{(0)} + h\,\mathbf{K} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + 0.1 \begin{bmatrix} 2 \\ -1 \\ 6.4142 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 1.9 \\ -0.3586 \end{bmatrix}$$

The value of $y'(0.1)$ is approximately $y_2^{(1)} = 1.9$.