

## Matrix properties

**The theory** (short excerpts from lectures)

### Vector norms:

A *vector norm* on  $R^n$  is a real valued function  $x \rightarrow \|x\|$  such that  $\forall x, y \in R^n, \forall \alpha \in R$  the following holds:

1.  $\|x\| \geq 0, \|x\| = 0 \Leftrightarrow x = 0$
2.  $\|\alpha x\| = |\alpha| \|x\|$
3.  $\|x + y\| \leq \|x\| + \|y\|$

*column norm:*  $\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$

*Euclidean norm:*  $\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2}$

*row norm:*  $\|x\|_\infty = \max(|x_1|, |x_2|, \dots, |x_n|)$

### Matrix norms:

A *p-norm*  $\|A\|_p$  of a matrix  $A$  (for  $p = 1, 2$  or  $\infty$ ) is a norm induced by the vector norm  $\|x\|_p$  as a maximal value of  $\|Ax\|_p$  on the unit sphere:

$$\|A\|_p = \max_{\|x\|_p=1} \|Ax\|_p$$

*Frobenius norm*  $\|A\|_F$  is a sum of all diagonal elements of  $A^T A$ :

$$\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2}$$

### Properties of square matrices:

*Spectral radius*  $\rho(A)$  of a matrix  $A$  is the maximum modulus of eigenvalues:

$$\rho(A) = \max_{i=1, \dots, n} |\lambda_i|, \quad \text{where } \lambda_i \text{ are eigenvalues of } A$$

*Trace*  $tr(A)$  of a matrix  $A$  is the sum of all its diagonal elements.

$A$  is *symmetric*, if  $A = A^T$ .

$A$  is *positive definite*, if  $x^T A x > 0 \forall x \neq 0$ .

*Condition number*  $\kappa(A)$  of  $A$  (relative to norm  $\|\cdot\|$ ) is  $\kappa(A) = \|A\| \cdot \|A^{-1}\|$ .

A matrix is *diagonally dominant* (abbreviation *d.d.*), if in every row the absolute value (or modulus) of the diagonal element is greater or equal than the sum of absolute values of all other elements in that row (or if this holds for  $A^T$ ).

A matrix is *strictly diagonally dominant* (abbreviation *s.d.d.*), if all these inequalities are strict.

**Theorems and lemmas:**

1.  $\|A\|_1 = \max(\sum_i |a_{i1}|, \sum_i |a_{i2}|, \dots, \sum_i |a_{in}|)$  – column norm
2.  $\|A\|_\infty = \max(\sum_j |a_{1j}|, \sum_j |a_{2j}|, \dots, \sum_j |a_{nj}|)$  – row norm
3.  $\|A\|_2 = \sqrt{\rho(A^T A)}$  – spectral norm
4.  $\|A\|_2 \leq \|A\|_F$
5.  $\|Ax\|_p \leq \|A\|_p \|x\|_p \quad \forall x$
6.  $\rho(A) \leq \|A\|$  for any norm of  $A$ .
7. Consistency of  $p$ -norms:  $\|AB\|_p \leq \|A\|_p \|B\|_p$ .  
Corollary:  $A^k$  converges to zero if any of its  $p$ -norms is less than 1.
8. General convergence result:  $A^k$  converges to zero  $\Leftrightarrow \rho(A) < 1$ .
9. If  $A$  is symmetric, then it has real eigenvalues and  $\|A\|_2 = \rho(A)$ .
10. If  $A$  is symmetric, then it is positive definite  $\Leftrightarrow$  all its leading principal minors are positive. (Sylvester's criterion)
11. If  $A$  is symmetric, then it is positive definite  $\Leftrightarrow$  all its eigenvalues are real positive numbers.

**Examples**

**Matrix and vector norms**

**Problem 1**

A matrix  $A$  and a vector  $y$  are given as

$$A = \begin{bmatrix} 2 & 0 \\ 3 & -4 \\ -3 & -2 \end{bmatrix} \quad y = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Compute all their norms (a vector is in fact a matrix of type  $n \times 1$ ).  
For the given matrix and vector, confirm the following inequality

$$\|Ay\| \leq \|A\| \cdot \|y\| \tag{1}$$

**The solution**

$$Ay = (4, 14, -2)^T$$

1-norm, or column norm (the maximum of the column sums):

$$\|A\|_1 = \max(|2| + |3| + |-3|, 0 + |-4| + |-2|) = \max(8, 6) = 8$$

$$\|y\|_1 = |2| + |-2| = 4$$

$$\|Ay\|_1 = |4| + |14| + |-2| = 20$$

$$20 \leq 8 \cdot 4 = 32$$

$\infty$ -norm, or row norm (the maximum of the row sums):

$$\|A\|_\infty = \max(|2| + 0, |3| + |-4|, |-3| + |-2|) = \max(2, 7, 5) = 7$$

$$\|y\|_\infty = \max(|2|, |-2|) = 2$$

$$\|Ay\|_\infty = \max(|4|, |14|, |-2|) = 14$$

$$14 \leq 7 \cdot 2 = 14$$

Frobenius norm:

$$\|A\|_F = \sqrt{2^2 + 0^2 + 3^2 + (-4)^2 + (-3)^2 + (-2)^2} = \sqrt{42} = 6.4807$$

( $\|A\|_2 \leq \|A\|_F$  - Frobenius norm is the upper limit, easier to compute)

$$\|y\|_2 = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2.8284$$

$$\|Ay\|_2 = \sqrt{4^2 + 14^2 + (-2)^2} = 14.6969$$

$$14.6969 \leq 6.4807 \cdot 2.8284 = 18.3300$$

Attention: the relationship (1) does not hold if different norms are mixed together! For instance, compare  $\|Ay\|_1$  and  $\|A\|_F, \|y\|_2$ .

**Spectral radius and spectral norm****Problem 2**

Compute spectral radius  $\rho(A)$  and spectral norm  $\|A\|_2$  of the matrix

$$A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$$

and confirm that  $\rho(A) \leq \|A\|$  holds for any norm of the matrix.

**The solution**

$$\det(A - \lambda I) = (-2 - \lambda)^2 + 1 = \lambda^2 + 4\lambda + 5 = 0 \Leftrightarrow \lambda_{1,2} = -2 \pm i$$

$$|\lambda_{1,2}| = \sqrt{(-2)^2 + (\pm 1)^2} = \sqrt{5}$$

$$\rho(A) = \max(|\lambda_1|, |\lambda_2|) = \max(\sqrt{5}, \sqrt{5}) = \sqrt{5} = 2.2361$$

$$\det(A^T \cdot A - \lambda I) = (5 - \lambda)^2 = 0 \Leftrightarrow \lambda_{1,2} = 5$$

$$\text{singular values } \sigma_{1,2} = \sqrt{5} = 2.2361 \Rightarrow \|A\|_2 = 2.2361$$

$$\|A\|_F = \sqrt{(-2)^2 + (-1)^2 + 1^2 + (-2)^2} = \sqrt{10} = 3.1623$$

$$\|A\|_1 = \|A\|_\infty = \max(|-2| + |-1|, |1| + |-2|) = \max(3, 3) = 3$$

$$2.2361 < 3.1623, \quad 2.2361 < 3.$$

## Symmetry, positive definiteness, diagonal dominance

### Problem 3

Consider the following matrices:

$$A = \begin{bmatrix} -5 & -1 & 0 \\ 3 & 3 & 1 \\ 1 & -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

- Which ones of them are symmetric?
- Which ones of them are strictly diagonally dominant?
- Which ones of them are symmetric positive definite?

### The solution

a) The matrices  $B$  and  $C$  are symmetric, the matrix  $A$  is not.

b) Matrix  $A$ :

$$\begin{array}{l} \text{rows:} \\ \begin{array}{l} |-5| > |-1| + |0| \\ |3| < |3| + |1| \\ |2| \geq |1| + |-1| \end{array} \end{array} \quad \begin{array}{l} \text{columns:} \\ \begin{array}{l} |-5| > |3| + |1| \\ |3| > |-1| + |-1| \\ |2| > |1| + |0| \end{array} \end{array}$$

The condition is violated at the second row  $\Rightarrow A$  is not d.d. by rows. Let us try the columns: the condition holds for all three columns, moreover, the inequalities are strict  $\Rightarrow A$  is s.d.d. by columns. It follows that  $A$  is strictly diagonally dominant.

The matrix  $B$  is symmetric - we can check the rows only (the columns would give the same result):

$$\begin{array}{l} |1| \geq |-1| + |0| \\ |2| \geq |-1| + |-1| \\ |2| > |0| + |-1| \end{array}$$

The condition holds for all rows, however the inequality is not always strict  $\Rightarrow B$  is diagonally dominant, although not strictly.

The matrix  $C$  is symmetric - we can check the rows only:

$$\begin{array}{l} |1| \geq |-1| + |0| \\ |1| < |-1| + |1| \\ |2| > |0| + |1| \end{array}$$

The matrix  $C$  is not d.d. by rows (neither by columns). It follows that the matrix  $C$  is not diagonally dominant.

c) A symmetric matrix is *positive definite*, if and only if all its leading principal minors are positive.

The matrix  $A$  is not symmetric.

The matrix  $B$ :

$$\det(1) = 1 > 0, \quad \det \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = 1 > 0, \quad \det \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = 1 > 0$$

All leading principal minors are positive and so the matrix  $B$  is positive definite.

The matrix  $C$ :

$$\det(1) = 1 > 0, \quad \det \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 0$$

The second leading principal minor is not positive, so we can stop investigating. The matrix  $C$  is not positive definite.

**Conclusions:** The matrix  $A$  is strictly diagonally dominant, it is not symmetric, nor positive definite. The matrix  $B$  is diagonally dominant (not strictly), it is symmetric and positive definite. The matrix  $C$  is symmetric, it is not diagonally dominant, nor positive definite.