

Iterative methods for linear systems

The theory (short excerpts from lectures)

Fixed-point iteration (abbreviation *fpi*)

The system $x = Ux + v$ is solved by the following algorithm:

1. choose $x^{(0)}$,
2. for $k = 0, 1, 2, \dots$ compute $x^{(k+1)} = Ux^{(k)} + v$, until $\|x^{(k+1)} - x^{(k)}\| < \epsilon$.

Convergence conditions:

$$\rho(U) < 1 \quad \Leftrightarrow \quad \textit{fpi} \text{ converges}$$

$$\|U\| < 1 \quad \Rightarrow \quad \textit{fpi} \text{ converges}$$

Solving the system $Ax = b$

The main idea: the system $Ax = b$ is transformed to a system $x = Ux + v$, which is then solved by *fpi*. The splitting $A = L + D + U$ is used, where L is the lower triangular part, D is the main diagonal and U is the upper triangular part.

Jacobi method (abbreviation *J*)

The system $Ax = b$ is expressed as $x = U_J x + v_J$, where $U_J = -D^{-1}(L + U)$ and $v_J = D^{-1}b$. This system is then solved by *fpi*.

Convergence conditions:

$$\rho(U_J) < 1 \quad \Leftrightarrow \quad J \text{ converges}$$

$$A \text{ is s.d.d.} \quad \Rightarrow \quad J \text{ converges}$$

Note: The eigenvalues λ_i of the matrix U_J can be computed from the equation $\det(L + \lambda D + U) = 0$, because

$$\det(U_J - \lambda I) = \det(-D^{-1}(L + U) - \lambda I) = \det(-D^{-1}) \cdot \det(L + \lambda D + U).$$

Gauss-Seidel method (abbreviation *GS*)

The system $Ax = b$ is expressed as $x = U_G x + v_G$, where $U_G = -(L + D)^{-1}U$ and $v_G = (L + D)^{-1}b$. This system is then solved by *fpi*.

Convergence conditions:

$$\rho(U_G) < 1 \quad \Leftrightarrow \quad GS \text{ converges}$$

$$A \text{ is s.d.d.} \quad \Rightarrow \quad GS \text{ converges}$$

$$A \text{ is symmetric and positive definite} \quad \Rightarrow \quad GS \text{ converges}$$

Note: The eigenvalues λ_i of the matrix U_G can be computed from the equation $\det(\lambda L + \lambda D + U) = 0$, because

$$\det(U_G - \lambda I) = \det(-(L + D)^{-1}U - \lambda I) = \det(-(L + D)^{-1}) \cdot \det(\lambda L + \lambda D + U).$$

Fixed-point iteration

Problem 1

Suppose the system $x = Ux + v$ is given as

$$U = \begin{bmatrix} 1/2 & 1 \\ -5/4 & -3/2 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

- a) Choose $x^{(0)} = (0, 0)^T$ and compute the first three iterations by *fpi*.
 b) Prove that *fpi* converges for the given system.

The solution

a)

$$x^{(1)} = Ux^{(0)} + v = \begin{bmatrix} 1/2 & 1 \\ -5/4 & -3/2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$x^{(2)} = Ux^{(1)} + v = \begin{bmatrix} 1/2 & 1 \\ -5/4 & -3/2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -5/2 \end{bmatrix}$$

$$x^{(3)} = Ux^{(2)} + v = \begin{bmatrix} 1/2 & 1 \\ -5/4 & -3/2 \end{bmatrix} \begin{bmatrix} 3 \\ -5/2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

b) First of all check the sufficient conditions, as they are easier to compute:

$$\|U\|_1 = \max(|1/2| + |-5/4|, |1| + |-3/2|) = \max(7/4, 10/4) = 10/4 > 1$$

$$\|U\|_\infty = \max(|1/2| + |1|, |-5/4| + |-3/2|) = \max(3/2, 11/4) = 11/4 > 1$$

$$\|U\|_F = \sqrt{(1/2)^2 + 1^2 + (-5/4)^2 + (-3/2)^2} = \sqrt{81/16} = 9/4 > 1$$

All norms are greater or equal than 1 and so we cannot conclude anything. Now we have to check the necessary and sufficient condition, it means to compute $\rho(U)$:

$$\det(U - \lambda I) = (1/2 - \lambda)(-3/2 - \lambda) + 5/4 = \lambda^2 + \lambda + 1/2 = 0$$

$$\Rightarrow \lambda_{1,2} = -1/2 \pm 1/2i$$

$$\|\lambda_{1,2}\| = \sqrt{(1/2)^2 + (\pm 1/2)^2} = \sqrt{1/2} = \sqrt{2}/2$$

$$\rho(U) = \max(\|\lambda_1\|, \|\lambda_2\|) = \sqrt{2}/2 < 1 \quad \Rightarrow \quad \textit{fpi} \text{ converges.}$$

Jacobi method

Problem 2

Consider following matrices:

$$A = \begin{bmatrix} -5 & -1 & 0 \\ 3 & 3 & 1 \\ 1 & -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

- a) Consider the system $Ax = b$ with $b = (2, 1, -1)^T$. Choose $x^{(0)} = (0, 0, 0)^T$ and compute the first two iterations by *J* method.
 b) Prove that *J* converges for the system $Ax = b$.
 c) Is there any easy-to-check condition to find out whether *J* converges for matrices *B* and *C*?

The solution

a) The iterations of J method are easier to compute by elements than in the vector form. The algorithm:

1. Write the system as equations:

$$\begin{aligned} -5x_1 - x_2 &= 2 \\ 3x_1 + 3x_2 + x_3 &= 1 \\ x_1 - x_2 + 2x_3 &= -1 \end{aligned}$$

2. Express the diagonal unknown from every equation:

$$\begin{aligned} x_1 &= -(2 + x_2)/5 & (1) \\ x_2 &= (1 - 3x_1 - x_3)/3 \\ x_3 &= (-1 - x_1 + x_2)/2 \end{aligned}$$

3. For $k = 0, 1, 2, \dots$ compute $x^{(k+1)}$ by substituting the elements $x^{(k)}$ from the previous iteration to the right hand side of (1):

$$\begin{aligned} x_1^{(k+1)} &= -(2 + x_2^{(k)})/5 \\ x_2^{(k+1)} &= (1 - 3x_1^{(k)} - x_3^{(k)})/3 \\ x_3^{(k+1)} &= (-1 - x_1^{(k)} + x_2^{(k)})/2 \end{aligned}$$

For $k = 0$ we have

$$\begin{aligned} x_1^{(1)} &= -(2 + x_2^{(0)})/5 = -(2 + 0)/5 = -2/5 \\ x_2^{(1)} &= (1 - 3x_1^{(0)} - x_3^{(0)})/3 = (1 - 0 - 0)/3 = 1/3 \\ x_3^{(1)} &= (-1 - x_1^{(0)} + x_2^{(0)})/2 = (-1 - 0 + 0)/2 = -1/2 \end{aligned}$$

The result of the first iteration is $x^{(1)} = (-2/5, 1/3, -1/2)^T$. The second iteration:

$$\begin{aligned} x_1^{(2)} &= -(2 + x_2^{(1)})/5 = -(2 + 1/3)/5 = -7/15 \\ x_2^{(2)} &= (1 - 3x_1^{(1)} - x_3^{(1)})/3 = (1 - 3(-2/5) - (-1/2))/3 = 9/10 \\ x_3^{(2)} &= (-1 - x_1^{(1)} + x_2^{(1)})/2 = (-1 - (-2/5) + 1/3)/2 = -2/15 \end{aligned}$$

The result of the second iteration is $x^{(2)} = (-7/15, 9/10, -2/15)^T$.

b) A is strictly diagonally dominant, which is a sufficient condition for convergence of J method.

c) Matrices B and C are not s.d.d., so the sufficient condition for convergence of J method is not fulfilled and so we do not know anything about the convergence. If we want to find out whether J method converges, we need to compute spectral radius of the matrix U_J . Generally, that is not easy for 3×3 matrices.

Problem 3

Consider any system $Ax = b$ with the following matrix and decide, whether J method converges:

$$A = \begin{bmatrix} 6 & 11 & -1 \\ 1 & 3 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

The solution

The matrix is not s.d.d., the sufficient convergence condition for J method does not hold and we have to compute spectral radius of the matrix U_J :

$$\det(U_J - \lambda I) = \begin{vmatrix} -\lambda & -\frac{11}{6} & \frac{1}{6} \\ -\frac{1}{3} & -\lambda & 0 \\ \frac{1}{2} & 0 & -\lambda \end{vmatrix} = -\lambda^3 + \frac{1}{12}\lambda + \frac{11}{18}\lambda = \lambda(-\lambda^2 + \frac{25}{36}) = 0$$

$$\Rightarrow \lambda_1 = 0, \lambda_{2,3} = \pm 5/6 \Rightarrow \rho(U_J) = 5/6 < 1.$$

Spectral radius of the matrix U_J is less than 1, so J method converges.

These eigenvalues can also be computed from the equation $\det(L + \lambda D + U) = 0$:

$$\begin{vmatrix} 6\lambda & 11 & -1 \\ 1 & 3\lambda & 0 \\ -1 & 0 & 2\lambda \end{vmatrix} = 36\lambda^3 - 3\lambda - 22\lambda = \lambda(36\lambda^2 - 25) = 0$$

Gauss-Seidel method

Problem 4

Consider the system $Ax = b$ given in Problem 2.

- a) Choose $x^{(0)} = (0, 0, 0)^T$ and compute the first two iterations by GS method.
- b) Prove that GS converges for the given system.
- c) Is there any easy-to-check condition to find out whether GS converges for matrices B and C given in Problem 2?

The solution

a) The iterations of GS method are easier to compute by elements than in the vector form. The algorithm:

1. Rewrite the system as equations - as in J method, see Problem 2.
2. From every equation, express the diagonal unknown - as in J method, see Problem 2.
3. For $k = 0, 1, 2, \dots$ compute $x^{(k+1)}$ one by one, starting with the first equation. The first element $x_1^{(k+1)}$ is computed as in J method, however, as soon as you compute a new unknown, substitute it immediately to the

right hand sides of all remaining equations of (1):

$$\begin{aligned}x_1^{(k+1)} &= -(2 + x_2^{(k)})/5 \\x_2^{(k+1)} &= (1 - 3x_1^{(k+1)} - x_3^{(k)})/3 \\x_3^{(k+1)} &= (-1 - x_1^{(k+1)} + x_2^{(k+1)})/2\end{aligned}$$

For $k = 0$ we have

$$\begin{aligned}x_1^{(1)} &= -(2 + x_2^{(0)})/5 = -(2 + 0)/5 = -2/5 \\x_2^{(1)} &= (1 - 3x_1^{(1)} - x_3^{(0)})/3 = (1 - 3(-2/5) - 0)/3 = 11/15 \\x_3^{(1)} &= (-1 - x_1^{(1)} + x_2^{(1)})/2 = (-1 - (-2/5) + 11/15)/2 = 1/15\end{aligned}$$

The result of the first iteration is $x^{(1)} = (-2/5, 11/15, 1/15)^T$. The second iteration:

$$\begin{aligned}x_1^{(2)} &= -(2 + x_2^{(1)})/5 = -(2 + 11/15)/5 = -41/75 \\x_2^{(2)} &= (1 - 3x_1^{(2)} - x_3^{(1)})/3 = (1 - 3(-41/75) - 1/15)/3 = 193/225 \\x_3^{(2)} &= (-1 - x_1^{(2)} + x_2^{(2)})/2 = (-1 - (-41/75) + 193/225)/2 = 91/450\end{aligned}$$

The result of the second iteration is $x^{(2)} = (-41/75, 193/225, 91/450)^T$.

b) A is strictly diagonally dominant, which is sufficient condition for convergence of GS method. GS method converges.

c) Check, that B is symmetric and positive definite, which is sufficient condition for convergence of GS method. The matrix C is not s.d.d. nor symmetric, and so neither condition sufficient for convergence of GS method is satisfied for the matrix C . Conclusions: GS method converges for the matrix B , nothing is known about convergence of GS method for matrix C . If we want to find out whether GS method converges for the matrix C , we need to compute spectral radius of the matrix U_G .

Problem 5

Consider the system $Ax = b$ from Problem 3 and decide, whether GS method converges.

The solution

The matrix A is not s.d.d. nor symmetric, and so neither of the sufficient convergence conditions for GS method hold and we have to compute the spectral radius of the matrix U_G . Computation of U_G is less straightforward than U_J , so we prefer to determine the eigenvalues from the equation $\det(\lambda(L+D)+U) = 0$:

$$\begin{aligned}\begin{vmatrix} 6\lambda & 11 & -1 \\ \lambda & 3\lambda & 0 \\ -\lambda & 0 & 2\lambda \end{vmatrix} &= 36\lambda^3 - 3\lambda^2 - 22\lambda^2 = \lambda^2(36\lambda - 25) = 0 \\ \Rightarrow \lambda_{1,2} &= 0, \lambda_3 = 25/36 \Rightarrow \rho(U_G) = 25/36 < 1.\end{aligned}$$

Spectral radius of the matrix U_G is less than 1, so GS method converges.