Iterative methods for linear systems

The theory (short excerpts from lectures)

Fixed-point iteration (abbreviation *fpi*)

The system $x = Ux + v$ is solved by the following algorithm:
1. choose $x^{(0)}$,
2. for $k = 0, 1, 2, \dots$ compute $x^{(k+1)} = Ux^{(k)} + v$, until $||x^{(k+1)} - x^{(k)}|| < \epsilon$.

Convergence conditions:
$\rho(U) < 1 \quad \Leftrightarrow \quad$ *fpi* converges
$||U|| < 1 \quad \Rightarrow \quad$ *fpi* converges

Solving the system Ax = b

The main idea: the system $Ax = b$ is transformed to a system $x = Ux + v$, which is then solved by *fpi*. The splitting $A = L + D + U$ is used, where L is the lower triangular part, D is the main diagonal and U is the upper triangular part.

Jacobi method (abbreviation J)

The system $Ax = b$ is expressed as $x = U_J\, x + v_J$, where $U_J = -D^{-1}(L + U)$ and $v_J = D^{-1}b$. This system is then solved by *fpi.*

Convergence conditions:
$\rho(U_J) < 1 \quad \Leftrightarrow \quad J$ converges
A is s.d.d. $\quad \Rightarrow \quad J$ converges

Gauss-Seidel method (abbreviation GS)

The system $Ax = b$ is expressed as $x = U_G\, x + v_G$, where $U_G = -(L + D)^{-1}U$ and $v_G = (L + D)^{-1}b$. This system is then solved by *fpi.*

Convergence conditions:
$\rho(U_G) < 1 \quad \Leftrightarrow \quad GS$ converges
A is s.d.d. $\quad \Rightarrow \quad GS$ converges
A is symmetric and positive definite $\quad \Rightarrow \quad GS$ converges

Fixed-point iteration

Problem 1

Suppose the system $x = Ux + v$ is given as

$$U = \begin{bmatrix} 1/2 & 1 \\ -5/4 & -3/2 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

a) Choose $x^{(0)} = (\,0,\,0)^T$ and compute the first three iterations by *fpi*.
b) Prove that *fpi* converges for the given system.

The solution

a)

$$x^{(1)} = Ux^{(0)} + v = \begin{bmatrix} 1/2 & 1 \\ -5/4 & -3/2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$x^{(2)} = Ux^{(1)} + v = \begin{bmatrix} 1/2 & 1 \\ -5/4 & -3/2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -5/2 \end{bmatrix}$$

$$x^{(3)} = Ux^{(2)} + v = \begin{bmatrix} 1/2 & 1 \\ -5/4 & -3/2 \end{bmatrix} \begin{bmatrix} 3 \\ -5/2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

b) First of all check the sufficient conditions, as they are easier to compute:
$||U||_1 = \max(\ |1/2| + |-5/4|,\ |\ 1\ | + |-3/2|) = \max(\ 7/4,\ 10/4) = 10/4 > 1$
$||U||_\infty = \max(\ |1/2| + |\ 1\ |,\ |-5/4| + |-3/2|) = \max(\ 3/2,\ 11/4) = 11/4 > 1$
$||U||_F = \sqrt{(1/2)^2 + 1^2 + (-5/4)^2 + (-3/2)^2} = \sqrt{81/16} = 9/4 > 1$
All norms are greater or equal than 1 and so we cannot conclude anything. Now we have to check the necessary and sufficient condition, it means to compute $\rho(U)$:

$\det(U - \lambda I) = (1/2 - \lambda)(-3/2 - \lambda) + 5/4 = \lambda^2 + \lambda + 1/2 = 0$
$\Rightarrow \quad \lambda_{1,2} = -1/2 \pm 1/2i$
$||\lambda_{1,2}|| = \sqrt{(1/2)^2 + (\pm 1/2)^2} = \sqrt{1/2} = \sqrt{2}/2$
$\rho(U) = \max(||\lambda_1||, ||\lambda_2||) = \sqrt{2}/2 < 1 \quad \Rightarrow \quad$ *fpi* converges.

Jacobi method

Problem 2

Consider following matrices:

$$A = \begin{bmatrix} -5 & -1 & 0 \\ 3 & 3 & 1 \\ 1 & -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

a) Consider the system $Ax = b$ with $b = (2, 1, -1)^T$. Choose $x^{(0)} = (\,0,\,0,\,0)^T$ and compute the first two iterations by J method.
b) Prove that J converges for the system $Ax = b$.
c) Is there any easy-to-check condition to find out whether J converges for matrices B and C?

The solution

a) The iterations of J method are easier to compute by elements than in the vector form. The algorithm:

1. Write the system as equations:

$$\begin{aligned} -5x_1 - x_2 \quad &= 2 \\ 3x_1 + 3x_2 + x_3 &= 1 \\ x_1 - x_2 + 2x_3 &= -1 \end{aligned}$$

2. Express the diagonal unknown from every equation:

$$\begin{aligned} x_1 &= -(2 + x_2)/5 \\ x_2 &= (1 - 3x_1 - x_3)/3 \\ x_3 &= (-1 - x_1 + x_2)/2 \end{aligned} \tag{1}$$

3. For $k = 0, 1, 2, \ldots$ compute $x^{(k+1)}$ by substituting the elements $x^{(k)}$ from the previous iteration to the right hand side of (1):

$$\begin{aligned} x_1^{(k+1)} &= -(2 + x_2^{(k)})/5 \\ x_2^{(k+1)} &= (1 - 3x_1^{(k)} - x_3^{(k)})/3 \\ x_3^{(k+1)} &= (-1 - x_1^{(k)} + x_2^{(k)})/2 \end{aligned}$$

For $k = 0$ we have

$$\begin{aligned} x_1^{(1)} &= -(2 + x_2^{(0)})/5 = -(2 + 0)/5 = -2/5 \\ x_2^{(1)} &= (1 - 3x_1^{(0)} - x_3^{(0)})/3 = (1 - 0 - 0)/3 = 1/3 \\ x_3^{(1)} &= (-1 - x_1^{(0)} + x_2^{(0)})/2 = (-1 - 0 + 0)/2 = -1/2 \end{aligned}$$

The result of the first iteration is $x^{(1)} = (-2/5,\, 1/3, -1/2)^T$. The second iteration:

$$\begin{aligned} x_1^{(2)} &= -(2 + x_2^{(1)})/5 = -(2 + 1/3)/5 = -7/15 \\ x_2^{(2)} &= (1 - 3x_1^{(1)} - x_3^{(1)})/3 = (1 - 3(-2/5) - (-1/2))/3 = 9/10 \\ x_3^{(2)} &= (-1 - x_1^{(1)} + x_2^{(1)})/2 = (-1 - (-2/5) + 1/3)/2 = -2/15 \end{aligned}$$

The result of the second iteration is $x^{(2)} = (-7/15,\, 9/10, -2/15)^T$.

b) A is strictly diagonally dominant, which is a sufficient condition for convergence of J method.

c) Matrices B and C are not s.d.d., so the sufficient condition for convergence of J method is not fulfilled and so we do not know anything about the convergence. If we want to find out whether J method converges, we need to compute spectral radius of the matrix U_J. Generally, that is not easy for 3×3 matrices.

Problem 3

Consider any system $Ax = b$ with the following matrix and decide, whether J method converges:

$$A = \begin{bmatrix} 6 & 11 & -1 \\ 1 & 3 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

The solution

The matrix is not s.d.d., the sufficient convergence condition for J method does not hold and we have to compute spectral radius of the matrix U_J:

$$det(U_J - \lambda I) = \begin{vmatrix} -\lambda & -\frac{11}{6} & \frac{1}{6} \\ -\frac{1}{3} & -\lambda & 0 \\ \frac{1}{2} & 0 & -\lambda \end{vmatrix} = -\lambda^3 + \frac{1}{12}\lambda + \frac{11}{18}\lambda = \lambda(-\lambda^2 + \frac{25}{36}) = 0$$

$$\Rightarrow \quad \lambda_1 = 0,\ \lambda_{2,3} = \pm 5/6 \quad \Rightarrow \quad \rho(U_J) = 5/6 < 1.$$

Spectral radius of the matrix U_J is less than 1, so J method converges.

Gauss-Seidel method

Problem 4

Consider the system $Ax = b$ given in Problem 2.
a) Choose $x^{(0)} = (\,0,\,0,\,0)^T$ and compute the first two iterations by GS method.
b) Prove that GS converges for the given system.
c) Is there any easy-to-check condition to find out whether GS converges for matrices B and C given in Problem 2?

The solution

a) The iterations of GS method are easier to compute by elements than in the vector form. The algorithm:

1. Rewrite the system as equations - as in J method, see Problem 2.
2. From every equation, express the diagonal unknown - as in J method, see Problem 2.
3. For $k = 0, 1, 2, \dots$ compute $x^{(k+1)}$ one by one, starting with the first equation. The first element $x_1^{(k+1)}$ is computed as in J method, however, as soon as you compute a new unknown, substitute it immediatelly to the right hand sides of all remaining equations of (1):

$$\begin{aligned} x_1^{(k+1)} &= -(2 + x_2^{(k)})/5 \\ x_2^{(k+1)} &= (1 - 3x_1^{(k+1)} - x_3^{(k)})/3 \\ x_3^{(k+1)} &= (-1 - x_1^{(k+1)} + x_2^{(k+1)})/2 \end{aligned}$$

For $k = 0$ we have

$$\begin{aligned}
x_1^{(1)} &= -(2 + x_2^{(0)})/5 = -(2+0)/5 = -2/5 \\
x_2^{(1)} &= (1 - 3x_1^{(1)} - x_3^{(0)})/3 = (1 - 3(-2/5) - 0)/3 = 11/15 \\
x_3^{(1)} &= (-1 - x_1^{(1)} + x_2^{(1)})/2 = (-1 - (-2/5) + 11/15)/2 = 1/15
\end{aligned}$$

The result of the first iteration is $x^{(1)} = (-2/5,\, 11/15,\, 1/15)^T$. The second iteration:

$$\begin{aligned}
x_1^{(2)} &= -(2 + x_2^{(1)})/5 = -(2 + 11/15)/5 = -41/75 \\
x_2^{(2)} &= (1 - 3x_1^{(2)} - x_3^{(1)})/3 = (1 - 3(-41/75) - 1/15)/3 = 193/225 \\
x_3^{(2)} &= (-1 - x_1^{(2)} + x_2^{(2)})/2 = (-1 - (-41/75) + 193/225)/2 = 91/450
\end{aligned}$$

The result of the second iteration is $x^{(2)} = (-41/75,\, 193/225, 91/450)^T$.

b) A is strictly diagonally dominant, which is sufficient condition for convergence of GS mehod. GS method converges.

c) Check, that B is symmetric and positive definite, which is sufficient condition for convergence of GS mehod. The matrix C is not s.d.d. nor symmetric, and so neither condition sufficient for convergence of GS method is satisfied for the matrix C. Conclusions: GS method converges for the matrix B, nothing is known about convergence of GS method for matrix C. If we want to find out whether GS method converges for the matrix C, we need to compute spectral radius of the matrix U_G.