

Fixed point iterations for linear systems (theory)

Fixed point iterations:

The goal: find the solution of a fixed point equation

$$X = F(X), \text{ where } X \in R^n, F : R^n \rightarrow R^n$$

The method: choose $X^{(0)}$, then for $k = 0, 1, 2, \dots$ compute

$$X^{(k+1)} = F(X^{(k)})$$

Theorem 1 – contraction mapping theorem:

Let $D \subset R^n$, D is closed, $F : D \rightarrow R^n$. Assume

1. if $X \in D$, then $F(X) \in D$
2. the mapping F is a *contraction* on D : there exists $q < 1$ such that

$$\|F(X) - F(Y)\| \leq q \|X - Y\| \quad \forall X, Y \in D$$

Then

- there exists unique $X^* \in D$ such that $X^* = F(X^*)$ and the fixed point iterations converge to X^* for any choice of $X^{(0)} \in D$,
- $X^{(k)}$ satisfies the a-priori error estimate $\|X^{(k)} - X^*\| \leq \frac{q^k}{1-q} \|X^{(1)} - X^{(0)}\|$
and the a-posteriori error estimate $\|X^{(k)} - X^*\| \leq \frac{q}{1-q} \|X^{(k)} - X^{(k-1)}\|$.

Proof: see [1]

Fixed point iterations for linear systems

Assume $F(X) = UX + V$, where U is a $n \times n$ matrix, $V \in R^n$, so the fixed point equation now represents a system of linear equations $X = UX + V$:

$$\begin{aligned} x_1 &= u_{11}x_1 + u_{12}x_2 + \dots + u_{1n}x_n + v_1 \\ x_2 &= u_{21}x_1 + u_{22}x_2 + \dots + u_{2n}x_n + v_2 \\ &\dots \\ x_n &= u_{n1}x_1 + u_{n2}x_2 + \dots + u_{nn}x_n + v_n \end{aligned}$$

which can be also rewritten in usual way as $(I - U)X = V$.

Under which assumptions the convergence of the fixed point iterations

$$X^{(k+1)} = U X^{(k)} + V$$

is guaranteed on R^n ?

From properties of any norm on R^n and its consistent matrix norm,

$$\|F(X) - F(Y)\| = \|UX + V - (UY + V)\| = \|UX - UY\| = \|U(X - Y)\| \leq \|U\| \|X - Y\|$$

holds $\forall X, Y \in R^n$, so the theorem follows:

Theorem 2 – sufficient condition for convergence of FPI:

If there exists a matrix norm such that $\|U\| < 1$, then the mapping $F(X) = UX + V$ is a contraction on R^n .

Now from Theorem 1 it follows that fixed point iterations $X^{(k+1)} = U X^{(k)} + V$ converge to the fixed point X^* for any choice of $X^{(0)}$. Moreover, both a-priori and a-posteriori error estimates hold with choice of $q = \|U\| < 1$ (provided the matrix norm and the vector norms are consistent).

What can be said about convergence of FPI, if there is no norm found for which $\|U\| < 1$?

Analysis of an error

Let $e^{(k)} = X^{(k)} - X^*$ is an error in k -th iteration of FPI. Then

$$e^{(k)} = X^{(k)} - X^* = (U X^{(k-1)} + V) - (U X^* + V) = U (X^{(k-1)} - X^*) = U e^{(k-1)}$$

$$e^{(k)} = U e^{(k-1)} = U^2 e^{(k-2)} = \dots = U^k e^{(0)}$$

Theorem 3 – necessary and sufficient condition for convergence of FPI:

The iteration process $X^{(k+1)} = U X^{(k)} + V$ converges to the fixed point X^* for *any* choice of $X^{(0)}$, if and only if $\rho(U) < 1$.

Proof: follows from the analysis of an error above and from the property $U^k \rightarrow 0 \Leftrightarrow \rho(U) < 1$. ([2] Th. 1.10)

Methods for solving $AX = B$ using fixed point iterations

Decompose given matrix A as $A = L + D + R$, where D is a diagonal matrix, L is a lower triangular and R is an upper triangular matrix. Assume that A has no zero elements on diagonal, so that inverse D^{-1} of D exists and also $(L + D)^{-1}$ exists.

Jacobi method

$$X = D^{-1}(L + R)X + D^{-1}B = U_J X + V_J$$

$$X^{(k+1)} = U_J X^{(k)} + V_J$$

Gauss-Seidel method

$$X = (L + D)^{-1}R X + (L + D)^{-1}B = U_G X + V_G$$

$$X^{(k+1)} = U_G X^{(k)} + V_G$$

Sufficient conditions (on matrix A) for convergence of FPI:

- Let A is strictly diagonally dominant.
Then both Jacobi and Gauss-Seidel methods converge for any $X^{(0)}$. ([2] Th. 4.9)
- Let A is symmetric positive definite.
Then Gauss-Seidel method converges for any $X^{(0)}$.
(see [2] Th. 4.10 – G-S is a special case of SOR for $\omega = 1$)

References

- [1] Tobias von Petersdorff: Contraction mapping theorem
<http://terpconnect.umd.edu/~petersd/666/fixedpoint.pdf>
- [2] Y. Saad: Iterative methods for sparse linear systems
http://www-users.cs.umn.edu/~saad/IterMethBook_2ndEd.pdf