Examples of theoretical problems, level B

Give reasons for all your answers.

1. Consider a linear system $x = Ux + v$, where

$$U = \begin{bmatrix} 0.5 & -0.6 \\ -0.5 & 0.2 \end{bmatrix}, \qquad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Will Fixed Point iterations converge for the given system?
- Give some sufficient conditions for convergence of Fixed Point iterations.
- What are necessary and sufficient conditions for convergence of Fixed Point iterations?
- Can any of the eigenvalues of U be equal to 1.5? (Give some reason for your answer other than computation of the eigenvalues).
- Give some upper limit on eigenvalues of U.

2. Consider a linear system $Ax = b$, where

$$A = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 6 \\ 6 \\ 12 \end{bmatrix}$$

- Is the matrix A diagonally dominant (or strictly d.d., or symmetric positive definite)?
- Will Gauss-Seidel iterative method converge for the given system?
- Will Jacobi iterative method converge for the given system?
- What are sufficient conditions (on matrix A) for convergence of Jacobi (or Gauss-Seidel) method?
- What are necessary and sufficient conditions for convergence of Jacobi (or Gauss-Seidel) method?

3. Consider the nonlinear system

$$\begin{aligned} x^2 + y^2 &= 4 \\ yx &= 1 \end{aligned}$$

- Derive the linear system which is to be solved in every step of Newton method for solution of a nonlinear system $F(x, y) = 0$, where $F = (f_1(x, y), f_2(x, y))^{\mathrm{T}}$.

4. Consider the following table of x_i and y_i coordinates of 5 points:

x_i	-1	-1	0	1	2
y_i	2.9	2.9	2.1	4	3.6

- Derive the system of equations for computing coefficients of the first degree polynomial $p_1(x)$, which approximates N given points $[x_i, y_i]$, $i = 1, \dots N$, using the least squares method.

- Write down the matrix form of the linear system of normal equations for computing the coefficients of the polynomial of the first (or second) degree.
- List some properties of the system matrix.

5. Consider Cauchy problem

$$y'' = x\,y' - \sqrt{y}\ , \qquad y(0) = 4, \quad y'(0) = 1\ .$$

- Find a domain G where the conditions of existence and uniqueness of the solution are satisfied.
- What are sufficient conditions for existence and uniqueness of solution for this problem?
- Show that substitution of the first forward (or backward) difference instead of the first derivative leads to $\mathcal{O}(h)$ error.
- What is the order of accuracy of explicit Euler method for the problem $y' = f(x, y),\ y(x_0) = y_0$?
- What is the order of accuracy of implicit Euler method for the problem $y' = f(x, y),\ y(x_0) = y_0$?
- What is the order of accuracy of Collatz method for the problem $y' = f(x, y),\ y(x_0) = y_0$?

6. Consider Cauchy problem

$$y'' = \frac{2}{x-7} + x\,y\ , \qquad y(0) = 0, \quad y'(0) = 2\ .$$

- Find an interval I of the maximal solution.
- What are sufficient conditions for existence and uniqueness of solution for this problem?

7. Consider a boundary value problem

$$y'' = \frac{2}{x-7} + x\,y\ , \qquad y(0) = 0, \quad y'(2) = 2\ .$$

- Prove that this problem has an unique solution on the given interval.
- What are the sufficient conditions for existence and uniqueness of solution of a boundary value problem $-(p(x)y'(x))' + q(x)y(x) = f(x),\ y(a) = \alpha,\ y(b) = \beta$?
- Derive the finite difference scheme for this equation (hint: use the central difference for approximation of y'').

8. Consider Dirichlet problem for Poisson equation

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = x - y$$

in the domain given by its vertices $A = [0,\ 0]$, $B = [1.5,\ 0]$, $C = [1,\ 1.5]$, $D = [0,\ 1.5]$ with prescribed value $u(x, y) = 2y$ on its boundary.

- Derive the finite difference scheme for the equation $-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x, y)$.

9. Consider mixed problem for heat equation

$$\frac{\partial u}{\partial t} = 0.3\,\frac{\partial^2 u}{\partial x^2} + x + t^2 \quad \text{in} \quad \Omega = \{[x,t] : x \in (0,1),\ t \in (0,T)\}\ ,$$

$u(x,0) = x^2, \quad u(0,t) = \sin(t), \quad u(1,t) = \frac{1}{2t+1}$.

- Derive the explicit scheme for the equation $\frac{\partial u}{\partial t} = p\frac{\partial^2 u}{\partial x^2} + f(x,t)$.
- Will the explicit method be stable for a choice of time step $\tau = 0.01$ and space step $h = 0.1$?

10. Consider mixed problem for wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\,\frac{\partial^2 u}{\partial x^2} + 2x - t \quad \text{in} \quad \Omega = \{[x,t] : x \in (0,1),\ t \in (0,T)\}\ ,$$

$u(x,0) = x^2, \quad u(0,t) = \sin(t), \quad u(1,t) = \frac{1}{2t+1}, \quad \frac{\partial u}{\partial t}u(x,0) = 1 - 2x$.

- Derive the explicit scheme for the equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + f(x,t)$.
- Will the explicit method be stable for a choice of time step $\tau = 0.01$ and space step $h = 0.1$?