Examples of theoretical problems, level A
(together with all theoretical problems for level B)
Give reasons for all your answers.

1. Consider a linear system $Ax = b$, where

$$A = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & 2 \\ 1 & 2 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 6 \\ 6 \\ 12 \end{bmatrix}$$

   - Can this system be solved using the Steepest descent method?
   - Derive Jacobi iterative method (in the matrix form $x^{i+1} = Ux^i + v$) for the system $Ax = b$.
   - Derive Gauss-Seidel iterative method (in the matrix form $x^{i+1} = Ux^i + v$) for the system $Ax = b$.

2. Consider Cauchy problem

$$y' = xy - \sqrt{y}\ , \qquad y(1) = 4\ .$$

   - Show, that the consistency error of explicit Euler method for the problem $y' = f(x, y),\ y(x_0) = y_0$ is $\mathcal{O}(h)$.
   - Show, that the local discretization error of explicit Euler method for the problem $y' = f(x, y),\ y(x_0) = y_0$ is $\mathcal{O}(h^2)$.
   - Under which assumptions substitution of the first central difference instead of the first derivative leads to $\mathcal{O}(h^2)$ error?

3. Consider a boundary value problem

$$y'' = \frac{2}{x-7} + x\,y\ , \qquad y(0) = 0, \quad y'(2) = 2\ .$$

   - Derive the finite difference scheme for a boundary value problem
     $-(p(x)\,y'(x))' + q(x)\,y(x) = f(x),\ y(a) = \alpha,\ y(b) = \beta$
     (hint: first, use the central difference with half the step for approximation of $z(x) = p(x)y'(x)$ at discretization points, second, use the central difference with half the step for approximation of $y'(x)$ at midpoints.

4. Consider Dirichlet problem for Poisson equation

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = x - y$$

   in the domain given by its vertices $A = [0,\ 0]$, $B = [1.5,\ 0]$, $C = [1,\ 1.5]$, $D = [0,\ 1.5]$ with prescribed value $u(x, y) = 2y$ on its boundary.

- Under which assumptions substitution of the second central difference instead of the second derivative leads to $\mathcal{O}(h^2)$ error?
- Show, that substitution of the second central difference instead of the second derivative leads to $\mathcal{O}(h^2)$ error.
- Derive the explicit scheme for the equation $-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x, y)$. What is the consistency error of this scheme, if only regular nodes are present?

5. Consider mixed problem for heat equation

$$\frac{\partial u}{\partial t} = 0.3 \frac{\partial^2 u}{\partial x^2} + x + t^2 \quad \text{in} \quad \Omega = \{[x, t] : x \in (0, 1),\ t \in (0, T)\}\ ,$$

$u(x, 0) = x^2, \quad u(0, t) = \sin(t), \quad u(1, t) = \frac{1}{2t+1}$.

- What is the consistency error of the explicit scheme?
- Derive the implicit scheme for the equation $\frac{\partial u}{\partial t} = p\frac{\partial^2 u}{\partial x^2} + f(x, t)$.
- Will the implicit method be stable for a choice of time step $\tau = 0.01$ and space step $h = 0.1$?
- Consider a given space step $h = 0.1$. From which interval the time step τ can be chosen, so that the explicit method is stable?
- Consider a given time step $\tau = 0.01$. From which interval the space step h can be chosen, so that the explicit method is stable?