Partial differential equations – classification

Second order linear PDE in 2 variables

We want to find a function $u \equiv u(x,y)$, which satisfies the equation

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + hu = f \qquad (1)$$

in a given domain Ω, where $a \equiv a(x,y), b \equiv b(x,y), \ldots h \equiv h(x,y)$ and $f \equiv f(x,y)$ are functions continuous on Ω. Moreover, we demand that $u(x,y)$ fulfills some given conditions on the boundary Γ of the domain Ω.

Classification of the 2-nd order linear PDE in 2 variables

The mathematical nature of the solutions of the equation (1) depends on the algebraic properties of the polynomial $ax^2 + bxy + cy^2 + dx + ey + q$; numerical method for solving the equation should be chosen accordingly to the type of the equation. The equations are classified by the sign of the discriminant
$r(x,y) = (b(x,y))^2 - 4\,a(x,y)\,c(x,y)$. There are three types of equations:

- **elliptic** $\ldots r(x,y) < 0$ (example: Poisson equation)
- **parabolic** $\ldots r(x,y) = 0$ (example: heat equation)
- **hyperbolic** $\ldots r(x,y) > 0$ (example: wave equation)

Problem

Consider the equation

$$x^2y^2\frac{\partial^2 u}{\partial x^2} - xy\frac{\partial^2 u}{\partial x \partial y} + 0.25\frac{\partial^2 u}{\partial y^2} = x + 2y$$

How is this equation classified?

Solution

We are interested in the sign of the discriminant
$r(x,y) = (b(x,y))^2 - 4\,a(x,y)\,c(x,y) = (-xy)^2 - 4\,x^2y^2 \cdot 0.25 = 0$.
Function $r(x,y)$ is equal to zero for all x, y, so the given equation is classified as parabolic (in any domain).